Event-triggered stabilization of linear systems under channel blackouts

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Allerton Conference, 30 Sept. 2015

Acknowledgements: National Science Foundation (Grants CNS-1329619, CNS-1446891)
Networked control systems

- Time-varying communication rates
- Channel may not be available during some intervals (blackouts)
- Time-triggered strategies would be very conservative
- Event-triggered controllers typically assume on-demand availability of channel\(^1\)

\(^1\) An important exception: Anta, Tabuada (2009)
Networked control systems

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Networked control systems

Shared communication resource

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- Quantization

  *Key to online state based transmission policy: data capacity*

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System description

**Plant dynamics:**
\[ \dot{x}(t) = A x(t) + B u(t), \quad u(t) = K \hat{x}(t), \quad x(t) \in \mathbb{R}^n \]
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# of bits transmitted at \( t_k \) is \( b_k = np_k \)

Can choose \( \{t_k\}, \{p_k\}, \{\tilde{r}_k\} \)
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Dynamic controller jump:
\[
\hat{x}(\tilde{r}_k) \triangleq q_k(x(t_k), \hat{x}(t_k^-))
\]

Encoding error: \( x_e \triangleq x - \hat{x} \)
Can design\textsuperscript{2} consistent algorithms for the encoder and decoder to implement quantizer \( q_k \) so that:

\begin{itemize}
  \item If the decoder knows \( d_e(t_0) \) s.t. \( \|x_e(t_0)\|_\infty \leq d_e(t_0) \)
  \item Both encoder and decoder compute recursively:
    \[ d_e(t) \triangleq \|e_{A}(t-t_k)\|_\infty \delta_k, \quad t \in [\tilde{r}_k, \tilde{r}_k+1) \]
    \[ \delta_{k+1} = \frac{1}{2} p_{k+1} \]
    \[ \text{then, } \|x_e(t)\|_\infty \leq d_e(t), \text{ for all } t \geq t_0 \]
\end{itemize}

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\]

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\delta_{k+1} = \frac{1}{2p_{k+1}} d_e(t_{k+1}).
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Lyapunov function: $x \mapsto V(x) = x^T P x$

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Lyapunov function: $x \mapsto V(x) = x^TPx$

Desired performance function: $V_d(t) = V_d(t_0)e^{-\beta(t-t_0)}$

Performance objective: ensure $h_{pf}(t) \triangleq \frac{V(x(t))}{V_d(t)} \leq 1$, for all $t \geq t_0$
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Design objective:

- Design event-triggered communication policy that is applicable to channels with time-varying rates and blackouts
- Recursively determine $\{t_k\}$, $\{p_k\}$ and $\{\tilde{r}_k\}$
- Ensure a uniform positive lower bound for $\{t_k - t_{k-1}\}_{k \in \mathbb{Z}_{>0}}$
Time-slotted channel model

\[ R(t) = R_j, \quad \forall t \in (\theta_j, \theta_{j+1}], \quad \text{min comm. rate: } \frac{p_k}{\Delta(t_k, p_k)} \geq R(t_k) \]

\[ \bar{p}(t) = \bar{\pi}_j, \quad \forall t \in (\theta_j, \theta_{j+1}], \quad \text{max packet size: } p_k \leq \bar{p}(t_k) \]

- \( j^{th} \) time-slot is of length \( T_j = \theta_{j+1} - \theta_j \)
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Need to quantify *data capacity*
max # of bits that can be *communicated* during the time interval \([\tau_1, \tau_2]\), overall all possible \(\{t_k\}\) and \(\{p_k\}\)

\[
D(\tau_1, \tau_2) \triangleq \max_{\{t_k\}, \{p_k\}} \ n \sum_{k=k_{\tau_1}}^{k_{\tau_2}} p_k
\]

Equivalent to optimal allocation of discrete # bits to be transmitted in each time slot.
Data capacity

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\text{max \# of bits that can be } \text{communicated} \text{ during the time interval } [\tau_1, \tau_2], \text{ overall all possible } \{t_k\} \text{ and } \{p_k\}
\]

\[
D(\tau_1, \tau_2) \triangleq \max \left\{ t_k \right\} \left\{ p_k \right\} \quad \text{s.t. } \ldots
\]

\[
\sum_{k=\tau_1}^{k=\tau_2} p_k = 3, \quad k_{\tau_1} = 3, \quad k_{\tau_2} = 7
\]

Equivalent to optimal allocation of \textit{discrete} \# bits to be transmitted in each time slot
Data capacity as allocation problem

Max # bits that may be transmitted in slot $j$

$$n_{\phi_j} \leq \begin{cases} 
  nR_j T_j + n\bar{\pi}_j, & \text{if } \bar{\pi}_j > 0 \\
  0, & \text{if } \bar{\pi}_j = 0 
\end{cases}$$

Available time in slot $j$ is affected by prior transmissions

Count only the bits also received

$$\phi_j \leq \begin{cases} 
  \bar{T}_j(\phi_j f_j) + \theta_j f_j - \theta_j + 1, & \text{if } \bar{T}_j(\phi_j f_j) > 0 \\
  0, & \text{otherwise} 
\end{cases}$$

$D(\theta_j f_j, \theta_j f_j) = \max_{\phi_j \in \mathbb{Z} \geq 0} \text{s.t. . . .}$
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$$n_{\phi_j} \leq \begin{cases} 
  nR_j\bar{T}_j(\phi_{j0}^j) + n\bar{\pi}_j, & \text{if } \bar{T}_j(\phi_{j0}^j) > 0 \\
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Available time in slot $j$ is affected by prior transmissions

$$n\phi_j \leq \begin{cases} nR_j\bar{T}_j(\phi_{j0}^f) + n\bar{\pi}_j, & \text{if } \bar{T}_j(\phi_{j0}^f) > 0 \\ 0, & \text{otherwise} \end{cases}$$

Count only the bits also received

$$\frac{\phi_j}{R_j} \leq \begin{cases} \bar{T}_j(\phi_{j0}^f) + \theta_{j+1} - \theta_j, & \text{if } \bar{T}_j(\phi_{j0}^f) > 0 \\ 0, & \text{otherwise.} \end{cases}$$
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$$D(\theta_{j0}, \theta_{j_f}) = \max_{\phi_j \in \mathbb{Z} \geq 0} \sum_{j=j_0}^{j_f-1} n \sum_{j=j_0}^{j_f-1} \phi_j.$$
A suboptimal solution for “slowly varying channels”

**Proposition**

Assume \( \frac{\bar{\pi}_j}{R_j} < T_{j+1}, \forall j \in N_{j_0}^j \) (any bits transmitted in slot \( j \) are received before the end of slot \( j + 1 \)).
A suboptimal solution for “slowly varying channels”

Proposition

Assume \( \frac{\bar{\pi}_j}{R_j} < T_{j+1}, \forall j \in \mathcal{N}_{j_0}^{j_f} \) (any bits transmitted in slot \( j \) are received before the end of slot \( j + 1 \)). Let \( \phi^r = \arg\max_{\phi_j \in \mathbb{R}_{\geq 0}} \sum_{j=j_0}^{j_f-1} \phi_j \) \((LP)\).

Let

\[ \phi^N \triangleq \lfloor \phi^r \rfloor \triangleq ([\phi^r_{j_0}], \ldots, [\phi^r_{j_f-1}]), \quad D_s(\theta_{j_0}, \theta_{j_f}) \triangleq n \sum_{j=j_0}^{j_f-1} \phi^N_j. \]
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Let

\[
\phi^N \triangleq \lfloor \phi^r \rfloor \triangleq (\lfloor \phi^r_{j_0} \rfloor, \ldots, \lfloor \phi^r_{j_f-1} \rfloor), \quad D_s(\theta_{j_0}, \theta_{j_f}) \triangleq n \sum_{j=j_0}^{j_f-1} \phi^N_j .
\]

Then

- \( \phi^N \) is a sub-optimal solution
- \( D(\theta_{j_0}, \theta_{j_f}) - D_s(\theta_{j_0}, \theta_{j_f}) \leq n(j_f - 1 - j_0) \).
Proposition

Let $\phi^*$ (or $\phi^N$) be any optimizing solution to $D(\theta_{j_0}, \theta_{j_f})$ (or $D_s(\theta_{j_0}, \theta_{j_f})$).
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Let $\phi^*$ (or $\phi^N$) be any optimizing solution to $D(\theta_{j_0}, \theta_{j_f})$ (or $D_s(\theta_{j_0}, \theta_{j_f})$). For any $t \in [\theta_{j_0}, \theta_{j_0+1})$ (any $t$ in $j_0$ slot)

$$\hat{D}(t, \theta_{j_f}) \triangleq [n \left[ \phi_{j_0}^* - R_{j_0}(t - \theta_{j_0}) \right]]_+ + n \sum_{j=j_0+1}^{j_f-1} \phi_j^*$$

$$\hat{D}_s(t, \theta_{j_f}) \triangleq [n \left[ \phi_{j_0}^N - R_{j_0}(t - \theta_{j_0}) \right]]_+ + n \sum_{j=j_0+1}^{j_f-1} \phi_j^N,$$
Proposition

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$$\hat{D}(t, \theta_{j_f}) \triangleq \left[ n \left[ \phi^*_{j_0} - R_{j_0}(t - \theta_{j_0}) \right] \right]_+ + n \sum_{j=j_0+1}^{j_f-1} \phi^*_j$$

$$\hat{D}_s(t, \theta_{j_f}) \triangleq \left[ n \left[ \phi^N_{j_0} - R_{j_0}(t - \theta_{j_0}) \right] \right]_+ + n \sum_{j=j_0+1}^{j_f-1} \phi^N_j,$$

Then, $0 \leq D(t, \theta_{j_f}) - \hat{D}(t, \theta_{j_f}) \leq n$ and $0 \leq D_s(t, \theta_{j_f}) - \hat{D}_s(t, \theta_{j_f}) \leq n$. 

Significance: Sufficient to solve the data capacity problem for intervals $[\theta_{j_0}, \theta_{j_f}]$ of interest.
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Recall performance objective: ensure $h_{pf}(t) \triangleq \frac{V(x(t))}{V_d(t)} \leq 1$, for all $t \geq t_0$
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Lemma

If \( h_{pf}(t) \leq 1 \) and \( h_{ch}(t) \leq 1 \)
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If $h_{pf}(t) \leq 1$ and $h_{ch}(t) \leq 1$ then $h_{pf}(s) \leq 1$, $\forall s \in [t, t + T']$. 
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- Make sure $h_{ch}(\tilde{r}_k) \leq 1$ so that future ability to control is not lost

$\tilde{L}_1(t) \triangleq \bar{h}_{pf}(\mathcal{T}(t), h_{pf}(t), \epsilon(t))$

$\tilde{L}_2(t) \triangleq \bar{h}_{ch}(\mathcal{T}(t), h_{pf}(t), \epsilon(t), \psi^{\tau_l}(t))$

$\mathcal{T}(t) \triangleq \begin{cases} T_M(\psi^{\tau_l}(t)), & \text{if } \psi^{\tau_l}(t) \geq 1 \\ \frac{2}{R(t)}, & \text{if } \psi^{\tau_l}(t) = 0. \end{cases}$
Role data capacity in control

\[ \tilde{L}_3(t) \equiv n \log_2 \left( e \bar{\mu} (\tau_l(t) - t) \epsilon(t) \epsilon_r(t) \right) - \sigma_1 \hat{D}_s(t, \tau_l(t)) \]

Transmission policy should be in tune with the optimal allocation

\[ \Phi_{\tau_l}(t) \equiv \left\lfloor P_j - R_j(t - \theta_j) \right\rfloor + , t \in (\theta_j, \theta_j + 1) \]

Artificial bound on packet size:

\[ \psi_{\tau_l}(t) \equiv \min\{ \bar{p}(t), \Phi_{\tau_l}(t) \} \]

If \( \tilde{L}_3(t_k) \leq 0 \) and \( p_k \leq \psi_{\tau_l}(t_k) \) then \( \tilde{L}_3(r_k) \leq 0 \)

If data capacity was "sufficient" at \( t_k \) and \( p_k \) respects artificial bound then data capacity is "sufficient" at \( r_k \)

But \( \psi_{\tau_l}(t) \) can be 0 when \( \bar{p}(t_k) > 0 \) (artificial blackouts)
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\[ \Phi^{\tau_l}(t) \triangleq [[\mathcal{P}_j - R_j(t - \theta_j)]]_+, \ t \in (\theta_j, \theta_{j+1}] \] (optim. alloc. in \((t, \theta_{j+1}]\))
Role data capacity in control

Transmission policy should be in tune with the optimal allocation

\[ \Phi_{\tau_l}(t) \triangleq \left[ [P_j - R_j(t - \theta_j)] \right]_+, \quad t \in (\theta_j, \theta_j+1) \] (optim. alloc. in \((t, \theta_j+1]\))

Artificial bound on packet size: \[ \psi_{\tau_l}(t) \triangleq \min\{\bar{p}(t), \Phi_{\tau_l}(t)\} \]
Role data capacity in control

\[ \tilde{L}_3(t) \triangleq n \log_2 \left( \frac{e^{\bar{\mu}(\tau_l(t)-t)\epsilon(t)}}{\epsilon_r(t)} \right) - \sigma_1 \hat{D}_s(t, \tau_l(t)) \]

Transmission policy should be in tune with the optimal allocation

\[ \Phi^{\tau_l}(t) \triangleq \lfloor P_j - R_j(t - \theta_j) \rfloor_+ , \ t \in (\theta_j, \theta_{j+1}] \ (\text{optim. alloc. in } (t, \theta_{j+1}]) \]

Artificial bound on packet size: \[ \psi^{\tau_l}(t) \triangleq \min\{\bar{p}(t), \Phi^{\tau_l}(t)\} \]

If \[ \tilde{L}_3(t_k) \leq 0 \] and \[ p_k \leq \psi^{\tau_l}(t_k) \]
If data capacity was “sufficient” at \[ t_k \] and \[ p_k \] respects artificial bound
Role data capacity in control

\[ \widetilde{L}_3(t) \triangleq n \log_2 \left( \frac{e^{\bar{\mu}(\tau_l(t) - t)} \epsilon(t)}{\epsilon_r(t)} \right) - \sigma_1 \hat{D}_s(t, \tau_l(t)) \]

Transmission policy should be in tune with the optimal allocation

\[ \Phi^{\tau_l}(t) \triangleq \lceil P_j - R_j(t - \theta_j) \rceil_+, \ t \in (\theta_j, \theta_{j+1}] \quad \text{(optim. alloc. in } (t, \theta_{j+1}] \text{)} \]

Artificial bound on packet size: \[ \psi^{\tau_l}(t) \triangleq \min\{\bar{p}(t), \Phi^{\tau_l}(t)\} \]

If \[ \widetilde{L}_3(t_k) \leq 0 \] and \[ p_k \leq \psi^{\tau_l}(t_k) \] then \[ \widetilde{L}_3(r_k) \leq 0 \]

If data capacity was “sufficient” at \( t_k \) and \( p_k \) respects artificial bound then data capacity is “sufficient” at \( r_k \)
Role data capacity in control

\[ \tilde{L}_3(t) \triangleq n \log_2 \left( \frac{e^{\bar{\mu}(\tau_l(t)-t)}\epsilon(t)}{\epsilon_r(t)} \right) - \sigma_1 \hat{D}_s(t, \tau_l(t)) \]

Transmission policy should be in tune with the optimal allocation

\[ \Phi^{\tau_l}(t) \triangleq \left[ \lfloor \mathcal{P}_j - R_j(t - \theta_j) \rfloor \right]_+, \quad t \in (\theta_j, \theta_j+1] \quad \text{(optim. alloc. in } (t, \theta_j+1]) \]

Artificial bound on packet size:

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If data capacity was “sufficient” at \( t_k \) and \( p_k \) respects artificial bound then data capacity is “sufficient” at \( r_k \)

But \( \psi^{\tau_l}(t) \) can be 0 when \( \bar{p}(t) > 0 \) (artificial blackouts)
Control policy in the presence of blackouts

\[ t_{k+1} = \min \left\{ t \geq \tilde{r}_k : \psi^{\tau_i}(t) \geq 1 \land \left( \max\{\tilde{L}_1(t), \tilde{L}_1(t^+), \tilde{L}_2(t), \tilde{L}_2(t^+)\} \geq 1 \right) \lor \max\{\tilde{L}_3(t), \tilde{L}_3(t^+)\} \geq 0 \right\}, \]
Control policy in the presence of blackouts

\[ t_{k+1} = \min \left\{ t \geq \tilde{r}_k : \psi^{\tau_l}(t) \geq 1 \land \left( \max\{\tilde{\mathcal{L}}_1(t), \tilde{\mathcal{L}}_1(t^+), \tilde{\mathcal{L}}_2(t), \tilde{\mathcal{L}}_2(t^+)\} \geq 1 \lor \max\{\tilde{\mathcal{L}}_3(t), \tilde{\mathcal{L}}_3(t^+)\} \geq 0 \right) \right\}, \]

\[ p_k \in \mathbb{Z}_{>0} \cap [\underline{p}_k, \psi^{\tau_l}(t_k)] \]

\[ \underline{p}_k \triangleq \min\{p \in \mathbb{Z}_{>0} : \overline{h}_{ch}(T_M(p), h_{pf}(t_k), \epsilon(t_k), p) \leq 1\}. \]
Control policy in the presence of blackouts

\[ t_{k+1} = \min \left\{ t \geq \tilde{r}_k : \psi^{\tau_l}(t) \geq 1 \land \right. \]
\[ \left. \left( \max\{ \tilde{\mathcal{L}}_1(t), \tilde{\mathcal{L}}_1(t^+), \tilde{\mathcal{L}}_2(t), \tilde{\mathcal{L}}_2(t^+) \} \geq 1 \right. \right. \]
\[ \left. \lor \max\{ \tilde{\mathcal{L}}_3(t), \tilde{\mathcal{L}}_3(t^+) \} \geq 0 \right\}, \]

\[ p_k \in \mathbb{Z}_{>0} \cap [p_k, \psi^{\tau_l}(t_k)] \]
\[ p_k \triangleq \min\{ p \in \mathbb{Z}_{>0} : \bar{h}_{ch}(T_M(p), h_{pf}(t_k), \epsilon(t_k), p) \leq 1 \}. \]

\[ \tilde{r}_k = \min\{ t \geq r_k : \psi^{\tau_l}(t) \geq 1 \lor p(t) = 0 \}. \]
Control policy in the presence of blackouts

Theorem

If

- $R(t) \geq \frac{(p+2)}{T_M(p)}$, $\forall p \in \{1, \ldots, p^{Max}\}$, $\forall t$
- $\tilde{L}_1(t_0) \leq 1$, $\tilde{L}_2(t_0) \leq 1$ and $\tilde{L}_3(t_0) \leq 0$ (initial feasibility)
- Conditions on blackout lengths

Then

- $\{t_k\}, \{p_k\}, \{\tilde{r}_k\}$ well defined
- Inter-transmission times have uniform positive lower bound
- $V(x(t)) \leq V_d(t_0) e^{-\beta(t-t_0)}$ for $t \geq t_0$ (origin is exponentially stable)
Control policy in the presence of blackouts

Theorem

If

1. \( R(t) \geq \frac{(p+2)}{T_M(p)}, \ \forall p \in \{1, \ldots, p^{Max}\}, \ \forall t \)
2. \( \tilde{L}_1(t_0) \leq 1, \tilde{L}_2(t_0) \leq 1 \) and \( \tilde{L}_3(t_0) \leq 0 \) (initial feasibility)
3. Conditions on blackout lengths

Then

1. \( \{t_k\}, \ \{p_k\}, \ \{\tilde{r}_k\} \) well defined
2. Inter-transmission times have uniform positive lower bound
3. \( V(x(t)) \leq V_d(t_0)e^{-\beta(t-t_0)} \) for \( t \geq t_0 \) (origin is exponentially stable)
Simulation results: 2D linear system

- # bits transmitted vs. time
- Channel communication rate vs. time
- Voltage vs. time
- Total # bits transmitted vs. time
Summary

Contribution:

- Fusion of event-triggered control and information-theoretic control
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• Fusion of event-triggered control and information-theoretic control

• Definition and computation of data capacity under full channel information

Future work:

• Address conservatism in the design

• Stochastic model of channel evolution

• Impact of the available information pattern at the encoder
Summary

Contribution:

- Fusion of event-triggered control and information-theoretic control
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