Distributed convergence to Nash equilibria by adversarial networks with directed topologies

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Abstract—This paper considers a class of strategic scenarios in which two cooperative groups of agents have opposing objectives with regards to the optimization of a common objective function. In the resulting zero-sum game, individual agents collaborate with neighbors in their respective network and have only partial knowledge of the state of the agents in the other network. We consider scenarios where the interaction topology within each cooperative network is given by a strongly connected and weight-balanced directed graph. We introduce a provably-correct distributed dynamics which converges to the set of Nash equilibria when the objective function is strictly concave-convex, differentiable, with globally Lipschitz gradient. The technical approach combines tools from algebraic graph theory, dynamical systems, convex analysis, and game theory.

I. INTRODUCTION

The nature of interactions between individual agents in a variety of networked scenarios is strategic and not necessarily cooperative. Examples of strategic interactions occur in biological systems, e.g., selfishness and stealth in collective motion [1] and competitive interactions between cells and organs [2], cybersecurity [3], and collective bargaining and opinion dynamics in heterogeneous networks [4], [5], [6]. This paper considers a class of such strategic scenarios where two networks of agents, with directed topologies and opposing goals, are involved in a zero-sum game, where the objective function is a sum of concave-convex functions. Within each network, agents cooperate with their neighbors and have partial information about the state of the agents of the opposing network. Our goal is to design a continuous-time distributed dynamics that can be used by the networks to converge to the set of Nash equilibria. Specifically, we seek to generalize the results of [7] to allow for directed topologies.

Literature review: This work is related to the literature on zero-sum games and distributed optimization. The convergence of the continuous-time best-response dynamics for zero-sum games with concave-convex payoff functions is shown in [8]. The results can be extended to quasiconvex-quasiconcave payoff functions, as recently shown in [9]. Continuous-time gradient flow dynamics has also been used for finding Nash equilibria of zero-sum games [10], [11]. This dynamics may fail to converge for general concave-convex functions [12] but is convergent when both convexity and concavity assumptions are strict. This convergence result also holds true when the payoff function is linear in one argument and its Hessian is positive-definite in the other [11], [12]. It is also worth noting that finding the saddle point of function using (sub)gradient dynamics has also been studied in discrete time [11], [13], [14]. The distributed computation of Nash equilibria in noncooperative games has been investigated in different contexts, see for example [15], [16], [17].

Regarding the literature on distributed optimization, the design of distributed dynamics for optimization of a sum of convex functions has been studied intensively in recent years, see e.g. [18], [19], [20]. These are consensus-based dynamics, see [21], [22], [23], [24], and are typically designed in discrete time. Exceptions are the works [25], [26], [7] on continuous-time distributed optimization on undirected networks and [27] on directed networks.

Statement of contributions: The contributions of this paper are threefold. We start by formulating a distributed zero-sum game for two networks with directed topologies engaged in a strategic scenario. The networks' objectives are to either maximize or minimize a common objective function which can be written as a sum of concave-convex functions. Individual agents collaborate with neighbors in their respective network and have partial knowledge of the state of the agents in the other network. We provide characterizations of the Nash equilibria of the game as saddle points of two newly-introduced functions that play a key role in the algorithm design. Secondly, we introduce a generalization of the saddle-point dynamics corresponding to these functions that also incorporates a design parameter. This strategy has a nice consensus plus gradient-based interpretation. Using the LaSalle Invariance Principle, we show that by appropriately choosing this parameter, the proposed dynamics asymptotically converges to the set of Nash equilibria for any pair of strongly connected weight-balanced adversarial networks and strictly concave-convex differentiable objective function with globally Lipschitz gradient. Interestingly, the interplay between the connectivity of the underlying networks and the Lipschitz constant of the gradient of the objective function plays a key role in determining the values of the design parameter. Finally, we provide a generalization to concave-convex functions of the known characterization of cocoercivity for concave functions, which plays a key role in our technical approach. The proofs are omitted for reasons of space and will appear elsewhere.

II. PRELIMINARIES

We start with some notational conventions. Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, $\mathbb{Z}_{\geq 1}$ denote the set of real, nonnegative real, integer, and positive integer numbers, respectively. We denote by $t: \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$, $d_1, d_2 \in \mathbb{Z}_{\geq 1}$, $d_2 \geq d_1$, any natural inclusion which maps each vector in $\mathbb{R}^{d_1}$ to a vector in $\mathbb{R}^{d_2}$ by adding
positively invariant set $S$ is closed and convex if it is concave in its first argument and convex in the second one [28]. A point $(x_1^*, x_2^*) \in X_1 \times X_2$ is a saddle point of $f$ if $f(x_1, x_2^*) \leq f(x_1^*, x_2^*) \leq f(x_1^*, x_2)$ for all $x_1 \in X_1$ and $x_2 \in X_2$. A function $f : \mathbb{R}^d \to \mathbb{R}$ is globally Lipschitz on $\mathbb{R}^d$ if for all $y, z \in \mathbb{R}^d$ there exists $C \in \mathbb{R}_{\geq 0}$ such that $|f(y) - f(z)| \leq C||y - z||$. For a differentiable function $f$, a point $x \in \mathbb{R}^d$ with $\nabla f(x) = 0$ is a critical point of $f$. A differentiable convex function $f$ satisfies, for all $x', x'' \in \mathbb{R}^d$, the first-order condition of convexity,

$$f(x') - f(x) \geq \nabla f(x) \cdot (x' - x). \quad (1)$$

### A. Stability analysis

Here, we recall some background on continuous-time dynamical systems following [29]. Consider a system on $X \subset \mathbb{R}^d$ given by

$$\dot{x}(t) = \Psi(x(t)), \quad (2)$$

where $t \in \mathbb{R}_{\geq 0}$ and $\Psi : X \subset \mathbb{R}^d \to \mathbb{R}^d$ is continuous. A solution to this dynamical system is a continuously differentiable curve $x : [0, T] \to X$ which satisfies (2). The set of equilibria of (2) is denoted by $\text{Eq}(\Psi) = \{x \in X | \Psi(x) = 0\}$.

The LaSalle Invariance Principle for continuous-time systems is helpful to establish the asymptotic stability properties of systems of the form (2). A set $W \subset X$ is positively invariant with respect to $\Psi$ if each solution with initial condition in $W$ remains in $W$ for all subsequent times. The Lie derivative of a continuously differentiable function $V : \mathbb{R}^d \to \mathbb{R}$ along $\Psi$ at $x \in \mathbb{R}^d$ is defined by $\mathcal{L}_\Psi V(x) = \nabla V(x) \cdot \Psi(x)$.

**Theorem 2.1:** (LaSalle Invariance Principle): Let $W \subset X$ be positively invariant under (2) and $V : X \to \mathbb{R}$ a continuously differentiable function. Suppose the evolutions of (2) with initial conditions in $W$ are bounded. Then any solution $x(t), t \in \mathbb{R}_{\geq 0}$, starting in $W$ converges to the largest positively invariant set $M$ contained in $S_{\Psi, V} \cap W$, where $S_{\Psi, V} = \{x \in X | \mathcal{L}_\Psi V(x) = 0\}$. When $M$ is a finite collection of points, then the limit of each solution equals one of them.

### B. Graph theory

We present some basic notions from algebraic graph theory following the exposition in [23]. A directed graph, or simply digraph, is a pair $G = (V, E)$, where $V$ is a finite set called the vertex set and $E \subseteq V \times V$ is the edge set. A digraph is undirected if $(v, u) \in E$ anytime $(u, v) \in E$. We refer to an undirected digraph as a graph. A path is an ordered sequence of vertices such that any ordered pair of vertices appearing consecutively is an edge of the digraph. A digraph is strongly connected if there is a path between any pair of distinct vertices. For a graph, we refer to this notion simply as connected. A weighted digraph is a triplet $G = (V, E, A)$, where $(V, E)$ is a digraph and $A \in \mathbb{R}^{n \times n}$ is the adjacency matrix of $G$, with the property that $a_{ij} > 0$ if $(v_i, v_j) \in E$ and $a_{ij} = 0$, otherwise. The weighted out-degree and in-degree of $v_i, i \in \{1, \ldots, n\}$, are respectively, $d_{\text{out}}^n(v_i) = \sum_{j=1}^n a_{ij}$ and $d_{\text{in}}^n(v_i) = \sum_{j=1}^n a_{ji}$. The weighted out-degree matrix $D_{\text{out}}$ is the diagonal matrix defined by $(D_{\text{out}})_{ii} = d_{\text{out}}^n(i)$, for all $i \in \{1, \ldots, n\}$. The Laplacian matrix is $L = D_{\text{out}} - A$. Note that $\lambda_n \leq 0$. If $G$ is strongly connected, then zero is a simple eigenvalue of $L$. $G$ is undirected if $L = L^T$ and weight-balanced if $d_{\text{out}}^n(v) = d_{\text{in}}^n(v)$, for all $v \in V$. Equivalently, $G$ is weight-balanced if and only if $\lambda_1^n L = 0$ if and only if $L + L^T$ is positive semidefinite. Furthermore, if $G$ is weight-balanced and strongly connected, then zero is a simple eigenvalue of $L + L^T$. Note that any undirected graph is weight-balanced.

### C. Zero-sum games

We recall some game theoretic notions from [30]. An $n$-player game is a triplet $G = (P, X, U)$, where $P$ is the set of players, $n = |P| \in \mathbb{Z}_{\geq 2}$, $X = X_1 \times \ldots \times X_n, X_i \subset \mathbb{R}^{d_i}$ is the set of (pure) strategies of player $v_i \in P, d_i \in \mathbb{Z}_{\geq 2}$, and $U = (u_1, \ldots, u_n)$, where $u_i : X \to \mathbb{R}$ is the payoff function of player $v_i, i \in \{1, \ldots, n\}$. The game $G$ is called a zero-sum game if $\sum_{i=1}^n u_i(x) = 0$, for all $x \in X$. If $x_i \in X_i$, we denote by $x_{-i}$ the strategy set of all players except $v_i$. An outcome $x^* \in X$ is called a (pure) Nash equilibrium of $G$ if for all $i \in \{1, \ldots, n\}$ and all $x_i \in X_i$ we have

$$u_i(x^*_i, x^-_i) \geq u_i(x_i, x^-_i).$$

One can extend this notion to mixed Nash equilibria by assignment of probabilities to pure strategies [30]. In this paper, we focus on a particular class of two-players zero-sum games which have at least one pure Nash equilibrium. The following well-known Minmax Theorem [31] characterizes that the game $G = \{(v_1, v_2), X_1 \times X_2, (u, -u)\}$ has a pure Nash equilibrium.

**Theorem 2.2:** (Minmax theorem): Let $X_1 \subset \mathbb{R}^{d_1}$ and $X_2 \subset \mathbb{R}^{d_2}, d_1, d_2 \in \mathbb{Z}_{\geq 2}$, be nonempty, closed, bounded, and convex. If $u : X_1 \times X_2 \to \mathbb{R}$ is continuous and the sets $\{x' \in X_1 | u(x', y) \geq \alpha \}$ and $\{x' \in X_2 | u(x, y') \leq \alpha \}$ are convex for all $x \in X_1, y \in X_2$, and $\alpha \in \mathbb{R}$, then

$$\min_{x} \max_{y} u(x, y) = \max_{y} \min_{x} u(x, y).$$

### III. Problem statement

Consider two networks $\Sigma_1$ and $\Sigma_2$ composed of agents $\{v_1, \ldots, v_{n_1}\}$ and agents $\{w_1, \ldots, w_{n_2}\}$, respectively. Throughout this paper, $\Sigma_1$ and $\Sigma_2$ are either connected undirected graphs, c.f. Section IV, or strongly connected weight-balanced digraphs, c.f. Section V. Since the latter case includes the first one, throughout this section, we assume the latter. The state of $\Sigma_1$, denoted by $x_1$, belongs to $X_1 \subset \mathbb{R}^{d_1}, d_1 \in \mathbb{Z}_{\geq 2}$. Likewise, the state of $\Sigma_2$, denoted by $x_2$, belongs to $X_2 \subset \mathbb{R}^{d_2}, d_2 \in \mathbb{Z}_{\geq 2}$. In this paper, we do not get into the details of what these states represent (as a particular case, the network state could correspond to the collection of the states of agents in it). In addition, each agent $v_i$ in $\Sigma_1$ has an estimate $x^*_i \in \mathbb{R}^{d_1}$ of what the network state
is, which may differ from the actual value $x_1$. Similarly, each agent $v_j$ in $\Sigma_2$ has an estimate $\hat{x}_j^2 \in \mathbb{R}^{d_2}$ of what the network state is. Within each network, neighboring agents can share their estimates. Networks can also obtain information about each other. This is modeled by means of a bipartite directed graph $\Sigma_{eng}$, called engagement graph, with disjoint vertex sets $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$, where every agent has at least one out-neighbor. According to this model, an agent in $\Sigma_1$ obtains information from its out-neighbors in $\Sigma_{eng}$ about their estimates of the state of $\Sigma_2$, and vice versa.

For each $i \in \{1, \ldots, n\}$, let $f_i^1 : X_1 \times X_2 \to \mathbb{R}$ be a locally Lipschitz concave-convex function only available to agent $v_i \in \Sigma_1$. Similarly, let $f_i^2 : X_1 \times X_2 \to \mathbb{R}$ be a locally Lipschitz concave-convex function only available to agent $w_j \in \Sigma_2$, $j \in \{1, \ldots, n\}$. The networks $\Sigma_1$ and $\Sigma_2$ are engaged in a zero-sum game with payoff function $U : X_1 \times X_2 \to \mathbb{R}$

$$U(x_1, x_2) = \sum_{i=1}^{n_1} f_i^1(x_1, x_2) = \sum_{j=1}^{n_2} f_j^2(x_1, x_2),$$

where $\Sigma_1$ wishes to maximize $U$, while $\Sigma_2$ wishes to minimize it. The objective of the networks is therefore to settle upon a Nash equilibrium, i.e., to solve the following maxmin problem

$$\max_{x_1 \in X_1} \min_{x_2 \in X_2} U(x_1, x_2).$$

We refer to this zero-sum game as the 2-network zero-sum game and denote it by $\Sigma_{eng}$.

We assume that $X_1 \subset \mathbb{R}^{d_1}$ and $X_2 \subset \mathbb{R}^{d_2}$ are compact convex. For convenience, let $x_1 = (x_1^1, \ldots, x_1^n_1)^T$ and $x_2 = (x_2^1, \ldots, x_2^n_2)^T$ denote vector of agent estimates about the state of the respective networks.

Remark 3.1: (Applications to distributed problems in the presence of adversaries): Multiple scenarios involving networked systems and intelligent adversaries in sensor networks, filtering, finance, and communications [32, 33] can be cast into the strategic framework described above. Here we present a class of examples from communications inspired by [34, Section 5.5.3]. Consider $n$ Gaussian communication channels, each with signal power $p_i \in \mathbb{R}_{>0}$ and noise power $\eta_i \in \mathbb{R}_{>0}$, for $i \in \{1, \ldots, n\}$. The capacity of each channel is proportional to $\log(1 + \beta p_i / (\sigma_i + \eta_i))$, where $\beta \in \mathbb{R}_{>0}$ and $\sigma_i > 0$ is the receiver noise. Note that capacity is concave in $p_i$ and convex in $\eta_i$. Both signal and noise powers must satisfy a budget constraint, i.e., $\sum_{i=1}^{n_1} p_i = P$ and $\sum_{i=1}^{n_1} \eta_i = C$, for some given $P, C \in \mathbb{R}_{>0}$. Two networks of $n$ agents are involved in this scenario, one, $\Sigma_1$, selecting signal powers to maximize capacity, the other one, $\Sigma_2$, selecting noise powers to minimize it. The network $\Sigma_1$ has decided that $m_1$ channels will have signal power $x_1$, while $n - 1 - m_1$ will have signal power $x_2$. The remaining $n - m_1$th channel has its power determined to satisfy the budget constraint, i.e., $P - m_1 x_1 - (n - 1 - m_1) x_2$. Likewise, the network $\Sigma_2$ does something similar with $m_2$ channels with noise power $y_1$, $n - 1 - m_2$ channels with noise power $y_2$, and one last channel with noise power $C - m_2 y_1 - (n - 1 - m_2) y_2$. Each network is aware of the partition made by the other

The individual objective function of the two agents (one from $\Sigma_1$, the other from $\Sigma_2$) making decisions on the power levels of the $i$th channel is the channel capacity itself. For $i \in \{1, \ldots, n - 1\}$, this takes the form

$$f_i(x, y) = \log \left(1 + \frac{\beta x_i}{\sigma_i + y_i}\right),$$

for some $\alpha, \beta \in \{1, 2\}$. Here $x = (x_1, x_2)$ and $y = (y_1, y_2)$. For $i = n$, it takes instead the form

$$f_n(x, y) = \log \left(1 + \frac{\beta(p - m_1 x_1 - (n - 1 - m_1) x_2)}{C - m_2 y_1 - (n - 1 - m_2) y_2}\right).$$

Note that $\sum_{i=1}^{n} f_i(x, y)$ is the total capacity of the $n$ communication channels.

A. Reformulation of the 2-network zero-sum game

In this section, we describe how agents in each network use the information obtained from their neighbors to compute the value of their own objective functions. Based on these estimates, we introduce a reformulation of the $\Sigma_{eng} = (\Sigma_1, \Sigma_2, \Sigma_{eng}, U)$, which is instrumental for establishing some of our results.

Each agent in $\Sigma_1$ has a locally Lipschitz, concave-convex function $\tilde{f}_i^1 : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$ with the properties:

- (Extension of own payoff function): for any $x_1 \in \mathbb{R}^{d_1}$, $x_2 \in \mathbb{R}^{d_2}$,

$$\tilde{f}_i^1(x_1, 1_{n_1} \otimes x_2) = f_i^1(x_1, x_2).$$

- (Projected over $\Sigma_{eng}$): there exists $\tilde{f}_i^1 : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$ such that, for any $x_1 \in \mathbb{R}^{d_1}$, $x_2 \in \mathbb{R}^{d_2}$,

$$\tilde{f}_i^1(x_1, x_2) = \tilde{f}_i^1(x_1, \pi_i^1(x_2)), \quad \text{(5b)}$$

with $\pi_i^1 : \mathbb{R}^{d_2} \to \mathbb{R}^{d_2} \Sigma_{eng}$ the projection of $x_2$ to the values received by $v_i$ from its out-neighbors in $\Sigma_{eng}$.

Each agent in $\Sigma_2$ has a function $\tilde{f}_i^2 : \mathbb{R}^{d_2} \times \mathbb{R}^{d_2} \to \mathbb{R}$ with similar properties. The collective payoff functions of the two networks are

$$\tilde{U}_1(x_1, x_2) = \sum_{i=1}^{n_1} \tilde{f}_i^1(x_1, x_2), \quad \text{(6a)}$$

$$\tilde{U}_2(x_1, x_2) = \sum_{j=1}^{n_2} \tilde{f}_j^2(x_1, x_2), \quad \text{(6b)}$$

In general, the functions $\tilde{U}_1$ and $\tilde{U}_2$ need not be the same. However, $\tilde{U}_1(1_{n_1} \otimes x_1, 1_{n_1} \otimes x_2) = \tilde{U}_2(1_{n_1} \otimes x_1, 1_{n_1} \otimes x_2)$, for any $x_1 \in \mathbb{R}^{d_1}$, $x_2 \in \mathbb{R}^{d_2}$. When both functions coincide, the next result shows that the original game can be lifted to a (constrained) zero-sum game.

Lemma 3.2: (Reformulation of the 2-network zero-sum game):Assume that the individual payoff functions $\{\tilde{f}_i^1\}_{i=1}^{n_1}$, $\{\tilde{f}_j^2\}_{j=1}^{n_2}$ satisfying (5) are such that the network payoff functions defined in (6) satisfy $\tilde{U}_1 = \tilde{U}_2$, and let $U$ denote this common function. Then, the problem (4) on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ is equivalent to the following problem on $\mathbb{R}^{n_1 d_1} \times \mathbb{R}^{n_2 d_2}$,

$$\max_{x_1 \in \mathbb{X}_1, x_2 \in \mathbb{X}_2} U(x_1, x_2),$$

subject to $\mathbf{L}_1 x_1 = 0_{n_1 d_1}$, $\mathbf{L}_2 x_2 = 0_{n_2 d_2}$, \quad \text{(7)}
with $L_\ell = L_\ell \otimes I_{d_\ell}$ and $L_\ell$ the Laplacian of $\Sigma_\ell$, $\ell \in \{1,2\}$. We denote by $G_{\text{adv-net}} = (\Sigma_1, \Sigma_2, \Sigma_{\text{eng}}, \tilde{U})$ the constrained zero-sum game defined by (7) and refer to this situation by saying that $G_{\text{adv-net}}$ can be lifted to $G_{\text{adv-net}}$. Our objective is to design a coordination algorithm that is implementable with the information that agents in $\Sigma_1$ and $\Sigma_2$ possess and leads them to find a Nash equilibrium of $G_{\text{adv-net}}$, which corresponds to a Nash equilibrium of $G_{\text{adv-net}}$ by Lemma 3.2. Achieving this goal, however, is nontrivial because individual agents, not networks themselves, are the decision makers. From the point of view of agents in each network, the objective is to agree on the states of both their own network and the other network, and that the resulting states correspond to a Nash equilibrium of $G_{\text{adv-net}}$.

We finish this section by presenting a characterization of the Nash equilibria of $G_{\text{adv-net}}$, instrumental for proving some of our upcoming results.

Proposition 3.3: (Characterization of the Nash equilibria of $G_{\text{adv-net}}$) For $\Sigma_1$, $\Sigma_2$, strongly connected and weight-balanced, define $F_1$ and $F_2$ by

$$F_1(x_1, z_1, x_2) = -\tilde{U}(x_1, x_2) + x_1^T L_1 z_1 + \frac{1}{2} x_1^T L_1 x_1,$$
$$F_2(x_2, z_2, x_1) = \tilde{U}(x_1, x_2) + x_2^T L_2 z_2 + \frac{1}{2} x_2^T L_2 x_2.$$

Then, $F_1$ and $F_2$ are convex in their first argument, linear in their second one, and concave in their third one. Moreover, assume $(x_1^*, z_1^*, x_2^*)$ satisfies the following saddle property for $(F_1, F_2)$: $(x_1^*, z_1^*)$ is a saddle point of $(x_1, z_1) \mapsto F_1(x_1, z_1, x_2^*)$ and $(x_2^*, z_2^*)$ is a saddle point of $(x_2, z_2) \mapsto F_2(x_2, z_2, x_1^*)$. Then,

(i) $(x_1^*, z_1^*, 1) \otimes a_1, x_2^*, z_2^*, 1 \otimes a_2)$ satisfies the saddle property for $(F_1, F_2)$ for any $a_1 \in \mathbb{R}^{d_1}$, $a_2 \in \mathbb{R}^{d_2}$, and

(ii) $(x_1^*, x_2^*)$ is a Nash equilibrium of $G_{\text{adv-net}}$.

Furthermore,

(iii) if $(x_1^*, x_2^*)$ is a Nash equilibrium of $G_{\text{adv-net}}$ then there exists $z_1^*, z_2^*$ such that $(x_1^*, z_1^*, x_2^*, z_2^*)$ satisfies the saddle property for $(F_1, F_2)$.

IV. DISTRIBUTED NASH SEEKING DYNAMICS FOR DIRECTED GRAPHS

Here, we review following [7] a dynamics which solves (7) when $\Sigma_1$ and $\Sigma_2$ are undirected. In this scenario, the gradi- ents of $F_1$ and $F_2$ are, respectively, distributed over $\Sigma_1$ and $\Sigma_2$. By Proposition 3.3, it is natural to consider the saddle-point dynamics for $F_1$ and $F_2$ to solve (4), i.e.,

$$\dot{x}_1 + L_1 x_1 + L_1 z_1 = \nabla_x \tilde{U}(x_1, x_2), \quad (8a)$$
$$\dot{z}_1 = L_1 x_1, \quad (8b)$$
$$\dot{x}_2 + L_2 x_2 + L_2 z_2 = -\nabla_x \tilde{U}(x_1, x_2), \quad (8c)$$
$$\dot{z}_2 = L_2 x_2, \quad (8d)$$

where $x_j, z_j \in \mathbb{R}^{n_1 d_j}, j \in \{1,2\}$. The following result establishes the convergence properties of this dynamics.

Theorem 4.1: (Asymptotic convergence of the directed distributed Nash seeking dynamics) Consider the zero-sum game $G_{\text{adv-net}} = (\Sigma_1, \Sigma_2, \Sigma_{\text{eng}}, \tilde{U})$, where

(i) $\Sigma_1$ and $\Sigma_2$ are connected and undirected,

(ii) $\tilde{U} : X_1^{n_1} \times X_2^{n_2} \to \mathbb{R}, X_1$ and $X_2$ compact convex subsets of, respectively, $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, is a differentiable strictly concave-convex function, distributed over $\Sigma_{\text{eng}}$ and also $\Sigma_1$ and $\Sigma_2$ in the sense of (6).

Then the projection onto the first and third components of the solutions to (8) asymptotically converges to the solution of (7).

It is worth mentioning that this result, in fact, also holds true when $\Sigma_1$ and $\Sigma_2$ are undirected and $U$ is the sum of locally Lipschitz concave-convex functions, see [7].

V. DISTRIBUTED NASH SEEKING DYNAMICS FOR DIRECTED GRAPHS

In this section, we introduce a continuous-time Nash seeking dynamics implementable over strongly connected and weight-balanced directed topologies. This dynamics is distributed over each individual network and can find the Nash equilibria of the zero-sum game, provided that the pay- off function is differentiable, strictly concave-convex, with globally Lipschitz gradient. This result generalizes the Nash seeking saddle-point dynamics of (8) to directed topologies.

We start by modifying the dynamics of (8) as

$$\dot{x}_1 + a L_1 x_1 + L_1 z_1 = \nabla_x \tilde{U}_{\epsilon}(x_1, x_2), \quad (9a)$$
$$\dot{z}_1 = L_1 x_1, \quad (9b)$$
$$\dot{x}_2 + a L_2 x_2 + L_2 z_2 = -\nabla_x \tilde{U}_{\epsilon}(x_1, x_2), \quad (9c)$$
$$\dot{z}_2 = L_2 x_2, \quad (9d)$$

where $a \in \mathbb{R}^{d_2}$ is a design parameter and the payoff function is differentiable with globally Lipschitz gradient. The reason behind including the parameter $a$ in the dynamics is that (8) may fail to converge when transcribed to directed graphs, for the same reason that the continuous-time saddle-point distributed optimization dynamics may fail on undirected graphs, see [27].

Next, we show that a suitable choice of this design parameter, makes this dynamics convergent.

Theorem 5.1: (Asymptotic convergence of the directed distributed Nash seeking dynamics) Consider the zero-sum game $G_{\text{adv-net}} = (\Sigma_1, \Sigma_2, \Sigma_{\text{eng}}, \tilde{U})$, where

(i) $\Sigma_1$ and $\Sigma_2$ are strongly connected and weight-balanced,

(ii) $\tilde{U} : X_1^{n_1} \times X_2^{n_2} \to \mathbb{R}, X_1$ and $X_2$ compact convex subsets of, respectively, $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, is a differentiable strictly concave-convex function with globally Lipschitz gradient, distributed over $\Sigma_{\text{eng}}$ and also $\Sigma_1$ and $\Sigma_2$, $\Sigma_{\text{eng}}$ in the sense of (6).

Let $h : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$h(r) = \frac{1}{2} \Lambda_{\min} \left( \frac{r^4 + 3r^2 + 2}{r} \right) + \sqrt{\left( \frac{r^4 + 3r^2 + 2}{r} \right)^2 - 4} + \frac{K r^2}{(1 + r^2)},$$

$$\Lambda_{\min} = \min_{j=1,2} \{ \Lambda_j (L_j + L_j^T) \},$$

where $\Lambda_j(\cdot)$ denotes the smallest non-zero eigenvalue and $K \in \mathbb{R}_{>0}$ is the Lipschitz
constant for the gradient of \( \hat{U} \). Then there exists \( \beta^* \in \mathbb{R}_{>0} \) with \( h_j(\beta^*) = 0, j \in \{1, 2\} \), such that for all \( 0 < \beta < \beta^* \), the projection onto the first and third components of the solutions of (9) with \( \alpha = \frac{\beta^2 + \beta}{\beta} \) asymptotically converges to the solution of (7).

Remark 5.2: (Comparison with the best-response dynamics): The advantage of using the gradient flow is that it avoids the cumbersome computation of the best-response map. This, however, does not come for free. There are concave-convex functions for which the (distributed) gradient flow dynamics, unlike the best-response dynamics, fails to converge to the saddle point, see [12] for an example.

Remark 5.3: (Scenarios with more than two adversarial networks): It is known that there are continuous-time zero-sum games with three players and strictly concave-convex payoff functions, for which even the best-response dynamics fails to converge, see [9]. This leaves little hope for extensions of Theorems 4.1 and 5.1 to \( N \)-network zero-sum games, with \( N \in \mathbb{Z}_{>2} \).

We finish this section with an example.

Example 5.4: (Distributed adversarial selection of signal and noise power via (9)): Recall the communication scenario described in Remark 3.1. Consider 5 channels, \( \{v_1, v_2, v_3, v_4, v_5\} \), for which the network \( \Sigma_1 \) has decided that \( \{v_1, v_3\} \) have signal power \( x_1 \) and \( \{v_2, v_4\} \) have signal power \( x_2 \). Channel \( v_5 \) has its signal power determined to satisfy the budget constraint \( P \in \mathbb{R}_{>0} \), i.e., \( P - 2x_1 - 2x_2 \). Similarly, the network \( \Sigma_2 \) has decided that \( v_1 \) has noise power \( y_1 \), \( \{v_2, v_3, v_4\} \) have noise power \( y_2 \), and \( v_5 \) has noise power \( C - y_1 - 3y_2 \) to meet the budget constraint \( C \in \mathbb{R}_{>0} \). We let \( \mathbf{x} = (x^1, x^2, x^3, x^4, x^5) \) and \( \mathbf{y} = (y^1, y^2, y^3, y^4, y^5) \), where \( x^i = (x^i_1, x^i_2) \in [0, P]^2 \) and \( y^i = (y^i_1, y^i_2) \in [0, C]^2 \), for each \( i \in \{1, \ldots, 5\} \).

The networks \( \Sigma_1 \) and \( \Sigma_2 \), which are weight-balanced and strongly connected, and the engagement topology \( \Sigma_{eng} \) are shown in Figure 1. Note that, according to this topology, each agent can observe the power employed by its adversary in its channel and, additionally, the agents in channel 2 can obtain information about the estimates of the opponent in channel 4 and vice versa. The payoff functions of the agents are given in Remark 3.1, where we take \( \sigma_1 = \sigma_3 \), for \( i \in \{1, 3, 5\} \), and \( \sigma_1 = \sigma_2 \), for \( i \in \{2, 4\} \), with \( \sigma_1, \sigma_2 \in \mathbb{R}_{>0} \).

This example fits into the approach described in Section III-A by considering the following extended payoff functions:

\[
\tilde{f}_1(x^1, y^1) = \log(1 + \frac{\beta x^1_1}{\sigma_1 + y^1_2}), \\
\tilde{f}_2(x^2, y^2) = \frac{1}{3} \log(1 + \frac{\beta x^2_1}{\sigma_2 + y^2_2}) + \frac{2}{3} \log(1 + \frac{\beta x^2_2}{\sigma_2 + y^2_2}), \\
\tilde{f}_3(x^3, y^3) = \log(1 + \frac{\beta x^3_1}{\sigma_1 + y^3_2}), \\
\tilde{f}_4(x^4, y^4) = \frac{1}{3} \log(1 + \frac{\beta x^4_1}{\sigma_2 + y^4_2}) + \frac{2}{3} \log(1 + \frac{\beta x^4_2}{\sigma_2 + y^4_2}), \\
\tilde{f}_5(x^5, y^5) = \log(1 + \frac{\beta(P - 2x^5_1 - 2x^5_2)}{\sigma_1 + C - y^5_1 - 3y^5_2}).
\]

\( \tilde{f}_1(x^1, y^1) = \tilde{f}_3(x^3, y^3) \) for the case study of Example 5.4 are shown. Edges which correspond to \( \Sigma_{eng} \) are dashed.

Fig. 1. The \( \Sigma_1, \Sigma_2 \) and \( \Sigma_{eng} \) for the case study of Example 5.4 are shown. Edges which correspond to \( \Sigma_{eng} \) are dashed.

VI. CONCLUSIONS AND FUTURE WORK

We have considered a class of strategic scenarios in which two networks of agents are involved in a zero-sum game. The networks’ objectives are to either maximize or minimize a common objective function. Individual agents collaborate with neighbors in their respective network and have partial knowledge of the state of the agents in the other network. Specifically, we have considered directed networks where information flows unidirectionally. We have introduced the directed distributed Nash-seeking dynamics and shown that, for appropriate parameter choices, this dynamics is guaranteed to converge to the Nash equilibrium for strictly concave-convex and differentiable objective functions with globally Lipschitz gradients. Future work will include relaxing the assumptions on the problem data under which convergence is guaranteed, including the smoothness, strict concavity-convexity properties, and sum decomposition of the objective function, and exploring the application of our results to various areas, including competitive social networks, collective bargaining, and collaborative pursuit-evasion.

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REFERENCES

Here, $y^0 = ((1, 0.5), (0.5, 1), (0.5, 0.5), (0.5, 1), (0.5, 1))^T$, $z^0 = 0_{10}$, $y^* = ((1, 0.5), (0.5, 1), (0.5, 0.5), (1, 0.5))^T$ and $z^* = 0_{10}$. The equilibrium $(x^*, z_1^*, y^*, z_2^*)$ is $x^* = (1.3471, 1.0315)^T \otimes 1_5$, $y^* = (1.5027, 0.3366)^T \otimes 1_5$, $z_1^* = (0.7508, 0.5084, 0.1447, 0.5084, 0.1447, -0.1271, -0.5201, -0.1271, -0.5201, -0.7626)^T$ and $z_2^* = (0.1079, -0.0987, -0.0002, 0.2237, 0.0358, 0.2875, -0.0360, 0.0087, -0.1076, -0.4213)$.


