s-Domain Circuit Analysis

Operate directly in the s-domain with capacitors, inductors and resistors

Key feature – linearity is preserved
Ccts described by ODEs and their ICs
Order equals number of C plus number of L

Element-by-element and source transformation
Nodal or mesh analysis for s-domain cct variables
Solution via Inverse Laplace Transform

Why?
1. Easier than ODEs
2. Easier to perform engineering design
3. Frequency response ideas - filtering
Element Transformations

Voltage source

Time domain

\[ v(t) = v_S(t) \]
\[ i(t) = \text{depends on cct} \]

Transform domain

\[ V(s) = V_S(s) = \mathcal{L}(v_S(t)) \]
\[ I(s) = \mathcal{L}(i(t)) \text{ depends on cct} \]

Current source

\[ I(s) = \mathcal{L}(i_S(t)) \]
\[ V(s) = \mathcal{L}(v(t)) \text{ depends on cct} \]
Element Transformations contd

Controlled sources

\[ v_1(t) = \mu v_2(t) \iff V_1(s) = \mu V_2(s) \]
\[ i_1(t) = \beta i_2(t) \iff I_1(s) = \beta I_2(s) \]
\[ v_1(t) = ri_2(t) \iff V_1(s) = rI_2(s) \]
\[ i_1(t) = gv_2(t) \iff I_1(s) = gV_2(s) \]

Short cct, open cct, OpAmp relations

\[ v_{SC}(t) = 0 \iff V_{SC}(s) = 0 \]
\[ i_{OC}(t) = 0 \iff I_{OC}(s) = 0 \]
\[ v_N(t) = v_P(t) \iff V_N(s) = V_P(s) \]

Sources and active devices behave identically

Constraints expressed between transformed variables

This all hinges on uniqueness of Laplace Transforms and linearity
Element Transformations contd

**Resistors**

\[ v_R(t) = R i_R(t) \quad i_R(t) = G v_R(t) \]
\[ V_R(s) = R I_R(s) \quad I_R(s) = G V_R(s) \]

**Capacitors**

\[ i_C(t) = C \frac{d v_C(t)}{d t} \]
\[ v_C(t) = \frac{1}{C} \int_0^t i_C(\tau) d\tau + v_C(0) \]
\[ I_C(s) = s C V_C(s) - C v_C(0-) \]
\[ V_C(s) = \frac{1}{sC} I_C(s) + \frac{v_C(0)}{s} \]

**Inductors**

\[ v_L(t) = L \frac{d i_L(t)}{d t} \]
\[ i_L(t) = \frac{1}{L} \int_0^t v_L(\tau) d\tau + i_L(0) \]
\[ V_L(s) = s L I_L(s) - L i_L(0) \]
\[ I_L(s) = \frac{1}{sL} V_L(s) + \frac{i_L(0)}{s} \]
\[ Z_L(s) = sL \quad Y_L(s) = \frac{1}{sL} \]
Element Transformations contd

Resistor

\[ i_R(t) \quad V_R(t) \quad R \quad I_R(s) \quad V_R(s) \quad R \]

\[ V_R(s) = R I_R(s) \]

Capacitor

\[ i_C(t) \quad v_C(t) \quad C \quad I_C(s) \quad V_C(s) \quad \frac{1}{sC} \quad V_C(0) \]

\[ I_C(s) = s CV_C(s) - C v_C(0) \]

\[ V_C(s) = \frac{1}{sC} I_C(s) + \frac{v_C(0)}{s} \]

Note the source transformation rules apply!
Element Transformations contd

**Inductors**

\[ V_L(s) = sIL_I(s) - L_i_L(0) \]
\[ I_L(s) = \frac{1}{sL}V_L(s) + \frac{i_L(0)}{s} \]

\[ v_L(t) \]
\[ i_L(t) \]

\[ V_L(s) \]
\[ I_L(s) \]
\[ + \]

\[ sL \]

\[ + \]
\[ L_L(0) \]

\[ - \]

\[ i_L(0) \]
**Example 10-1, T&R, 5th ed, p 456**

**RC cct behavior**

Switch in place since \( t = -\infty \), closed at \( t = 0 \). Solve for \( v_C(t) \).

**Initial conditions**

\[
v_C(0) = V_A
\]

**s-domain solution using nodal analysis**

\[
I_1(s) = \frac{V_C(s)}{R} \quad I_2(s) = \frac{V_C(s)}{sC} = sCV_C(s)
\]

**t-domain solution via inverse Laplace transform**

\[
V_C(s) = \frac{V_A}{s + \frac{1}{RC}} \quad v_c(t) = V_A e^{-\frac{t}{RC}} u(t)
\]
Example 10-2 T&R, 5th ed, p 457

Solve for $i(t)$

\[
\begin{align*}
\text{KVL around loop} & \quad \frac{V_A}{s} - (R + sL)I(s) + Li_L(0) = 0 \\
\text{Solve} & \quad I(s) = \frac{V_A}{s} + \frac{i_L(0)}{s + \frac{R}{L}} + \frac{\left( i_L(0) - \frac{V_A}{R} \right)}{s + \frac{R}{L}} \\
\text{Invert} & \quad i(t) = \left[ \frac{V_A}{R} - \frac{V_A}{R} e^{-\frac{Rt}{L}} + i_L(0)e^{-\frac{Rt}{L}} \right] u(t) \text{ Amps}
\end{align*}
\]
Impedance and Admittance

**Impedance** ($Z$) is the $s$-domain proportionality factor relating the transform of the voltage across a two-terminal element to the transform of the current through the element with all initial conditions zero

$$V(s) = Z(s)I(s)$$

**Admittance** ($Y$) is the $s$-domain proportionality factor relating the transform of the current through a two-terminal element to the transform of the voltage across the element with initial conditions zero

$$I(s) = Y(s)V(s)$$

Impedance is like resistance
Admittance is like conductance
Circuit Analysis in s-Domain

Basic rules

The equivalent **impedance** $Z_{eq}(s)$ of two impedances $Z_1(s)$ and $Z_2(s)$ in series is $Z_{eq}(s) = Z_1(s) + Z_2(s)$

Same current flows

\[ V(s) = Z_1(s)I(s) + Z_2(s)I(s) = Z_{eq}(s)I(s) \]

\[ I(s) = Y_1(s)V(s) + Y_2(s)V(s) = Y_{eq}(s)V(s) \]

The equivalent **admittance** $Y_{eq}(s)$ of two admittances $Y_1(s)$ and $Y_2(s)$ in parallel is $Y_{eq}(s) = Y_1(s) + Y_2(s)$
Example 10-3 T&R, 5th ed, p 461

Find $Z_{AB}(s)$ and then find $V_2(s)$ by voltage division

$Z_{eq}(s) = sL + R \parallel \frac{1}{sC} = sL + \frac{1}{\frac{1}{R} + sC} = \frac{RLCs^2 + Ls + R}{RCs + 1} \Omega$

$V_2(s) = \left[ \frac{Z_1(s)}{Z_{eq}(s)} \right] V_1(s) = \left[ \frac{R}{RLCs^2 + sL + R} \right] V_1(s)$
Example

Formulate node voltage equations in the $s$-domain

\[
\begin{align*}
\mathbf{R}_1 v_1(t) &+ \mathbf{R}_2 C_2 v_2(t) + \mathbf{R}_3 v_x(t) + \mu v_x(t) = \mathbf{V}_1(s) \\
\mathbf{R}_1 V_1(s) &+ \mathbf{R}_2 C_2 v_2(s) + \mathbf{R}_3 V_x(s) + \mu V_x(s) = \mathbf{V}_2(s)
\end{align*}
\]
Example contd

Node A: $V_A(s) = V_1(s)$

Node B:
$$
\frac{V_B(s) - V_A(s)}{R_1} + \frac{V_B(s) - V_D(s)}{R_2} + \frac{V_B(s)}{1/sC_1} + \frac{V_B(s) - V_C(s)}{1/sC_2} - C_1v_{C1}(0) - C_2v_{C2}(0) = 0
$$

Node C:
$$
-sC_2V_B(s) + \left[ sC_2 + G_3 \right]V_C(s) = -C_2v_{C2}(0)
$$

Node D: $V_D(s) = \mu V_x(s) = \mu V_C(s)$
Example

Find $v_O(t)$ when $v_S(t)$ is a unit step $u(t)$ and $v_C(0)=0$

Convert to $s$-domain
Example

Nodal Analysis

Node A: \( V_A(s) = V_S(s) \)

Node D: \( V_D(s) = V_O(s) \)

Node C: \( V_C(s) = 0 \)

Node B: \( (G_1 + sC)V_B(s) - G_1V_S(s) = C_vC(0) \)

Node C KCL: \( -sCV_B(s) - G_2V_O(s) = -C_vC(0) \)

Solve for \( V_O(s) \)

\[
V_O(s) = -\left[ \frac{sG_1C}{G_1 + sC} \right] V_S(s) = -\left[ \frac{R_2}{R_1} \times \frac{s}{s + \frac{1}{R_1C}} \right] V_S(s)
\]

\[
= -\left[ \frac{R_2}{R_1} \times \frac{s}{s + \frac{1}{R_1C}} \right] \frac{1}{s} = \frac{-R_2}{R_1} \times \frac{1}{s + \frac{1}{R_1C}}
\]

Invert LT

\[
v_O(t) = \frac{-R_2}{R_1} e^{-\frac{R_1C}{t}} u(t)
\]
Superposition in s-domain ccts

The s-domain response of a cct can be found as the sum of two responses

1. The zero-input response caused by initial condition sources, with all external inputs turned off
2. The zero-state response caused by the external sources, with initial condition sources set to zero

Linearity and superposition

Another subdivision of responses

1. Natural response – the general solution
   Response representing the natural modes (poles) of cct
2. Forced response – the particular solution
   Response containing modes due to the input
Example 10-6, T&R, 5th ed, p 466

The switch has been open for a long time and is closed at t=0.

Find the zero-state and zero-input components of V(s).

Find v(t) for $I_A = 1\text{mA}$, $L = 2\text{H}$, $R = 1.5\text{K}\Omega$, $C = 1/6 \mu\text{F}$
Example 10-6 contd

\[ V_{zs}(s) = Z_{eq}(s) \frac{I_A}{s} = \frac{I_A}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \]

\[ V_{zi}(s) = Z_{eq}(s) RCI_A = \frac{RI_A}{s} \]

Substitute values

\[ V_{zs}(s) = \frac{6000}{(s + 1000)(s + 3000)} = \frac{3}{s + 1000} + \frac{-3}{s + 3000} \]

\[ v_{zs}(t) = \left[ 3e^{-1000t} - 3e^{-3000t} \right] u(t) \]

\[ V_{zi}(s) = \frac{1.5s}{(s + 1000)(s + 3000)} = \frac{-0.75}{s + 1000} + \frac{2.25}{s + 3000} \]

\[ v_{zi}(t) = \left[ -0.75e^{-1000t} + 2.25e^{-3000t} \right] u(t) \]

What are the natural and forced responses?
Features of s-domain cct analysis

The response transform of a finite-dimensional, lumped-parameter linear cct with input being a sum of exponentials is a rational function and its inverse Laplace Transform is a sum of exponentials. The exponential modes are given by the poles of the response transform. Because the response is real, the poles are either real or occur in complex conjugate pairs. The **natural modes** are the zeros of the cct determinant and lead to the natural response. The **forced poles** are the poles of the input transform and lead to the forced response.
Features of s-domain cct analysis

A cct is **stable** if all of its poles are located in the open left half of the complex s-plane

A key property of a system

**Stability:** the natural response dies away as $t \to \infty$

Bounded inputs yield bounded outputs

A cct composed of $R_s$, $C_s$ and $L_s$ will be at worst marginally stable

With $R_s$ in the right place it will be stable

$Z(s)$ and $Y(s)$ both have no poles in $\text{Re}(s) > 0$

Impedances/admittances of RLC ccts are “Positive Real” or energy dissipating