Lecture 1: introduction

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Abstract

This lecture states the goals of the course and provides an introduction to basic concepts. We discuss general dynamical properties of nonlinear systems and explore its differences with linear systems. We also introduce some interesting nonlinear examples. Finally, we focus on the class of planar systems, for which we discuss periodic orbits and bifurcations. The treatment corresponds to Chapters 1-2 in [1] and Chapters 1-2 in [2].

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1 Introduction to nonlinear dynamical systems

1.1 Analysis versus simulation

Computers and software are getting better and better at simulating complex dynamical systems. Simulation coupled with good intuition can provide useful insight into a system’s behavior. Nevertheless, it is impossible to rely entirely on simulation to study properties such as stability, reachability, controllability, etc, since crucial cases may be overlooked or ill-stated.

Analysis tools provide us with formal mathematical proofs about a system’s behavior. They also give us the necessary insight to design better engineering systems. Many times, unexpected phenomena arise we have not thought of simulating. Furthermore, fundamental properties such as stability, reachability and observability cannot be reliably assessed only through simulations.

1.2 Linear versus nonlinear

There is a huge body of work in the analysis and control of linear systems, which are of the form

\[ \dot{x}(t) = A(t)x(t) + B(t)u, \quad x(0) = x_0 \]

where \( x \in \mathbb{R}^n \) (the state), \( u \in \mathbb{R}^m \) (the control inputs), \( A(t) \in \mathbb{R}^{n \times n} \) (the dynamics) and \( B(t) \in \mathbb{R}^{n \times m} \) (the control action).

Most models currently available are linear - specially in industry. However, most real systems are nonlinear… Why nonlinear theory? Because the dynamics of linear systems are not rich enough to describe many commonly observed phenomena like

- **Multiple (finite) equilibria.** Consider the linear differential equation

\[ \dot{x}(t) = A x(t) \]

For the initial condition \( x_0 \in \mathbb{R}^n \), the solution of the system is \( x(t) = \exp(At)x_0 \).

The point \( x = 0 \) is an equilibrium. If \( A \) is non-singular, then \( x = 0 \) is the only equilibrium of the system. If \( A \) is singular, then \( \ker A \) is the set of equilibria. However, an infinitesimal perturbation makes “almost surely” non-singular any singular matrix.

Therefore, problems in Nature and engineering which have a finite number of equilibria cannot be described by means of a linear system model.

*Example:* a pendulum
1.3 Goals of this course

(i) To learn and use various notions and tools for the analysis and control of nonlinear systems.

(ii) To get a feeling and gain insight into the complexity of nonlinear systems.

(iii) To introduce a wide variety of interesting, inherently nonlinear examples.

2 Basic concepts

Basic notions from differentiable topology on $\mathbb{R}^n$ that by now you should be familiar with are the following:

- **B(x, δ)** denotes a ball centered at $x$ of radius $δ$, i.e., $B(x, δ) = \{y \in \mathbb{R}^n \mid ||y - x||_2 < δ\}$
- a **neighborhood** $N$ of a point $x \in \mathbb{R}^n$ is a set such that there exists $δ > 0$ with $B(x, δ) \subset N$
- a **limit point** $x \in \mathbb{R}^n$ of a set $S \subset \mathbb{R}^n$ is a point such that every neighborhood of $x$ contains a point of $S \backslash \{x\}$
- a closed set \( S \) is a set that contains all its limit points. A set is closed if and only if it contains its boundary.
- an open set \( S \) is a set such that for any \( x \in S \), there exists a neighborhood of \( x \) contained in \( S \).
- a bounded set \( S \) is such that there exists \( K > 0 \) such that \( \| x \| \leq K \) for all \( x \in S \).
- a compact set \( S \) is a bounded and closed set.
- a connected set \( S \) is such that for all \( U, V \) open with \( U \cap V = \emptyset \) and \( S \subset U \cup V \), then either \( S \cap U = \emptyset \) or \( S \cap V = \emptyset \). Otherwise said, one cannot partition a connected set.
- a region or a domain is an open connected set.

Consider the dynamical system
\[
\dot{x} = f(x, t),
\]
where \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \). Let us define the following notions:

(i) the system is called autonomous if \( f \) does not explicitly depend on time, i.e., \( f(x, t) = f(x) \);

(ii) a point \( x_0 \in \mathbb{R}^n \) is called an equilibrium point if \( f(x_0, t) = 0 \) for all \( t \);

(iii) an equilibrium \( x_0 \) is called isolated if there exists \( \delta > 0 \) such that there is no other equilibrium point in the ball \( B(x_0, \delta) \).

Let \( x_0 \) be an equilibrium of the autonomous system \( \dot{x} = f(x) \). Assume \( f \) is \( C^1 \). Writing down the Taylor series about \( x_0 \), we get
\[
f(x) = f(x_0) + \frac{\partial f}{\partial x}(x_0)(x - x_0) + r(x) = \frac{\partial f}{\partial x}(x_0)(x - x_0) + r(x),
\]
where \( r(x) \) is the tail of the Taylor series corresponding to the higher-order terms, i.e.,
\[
\lim_{\|x - x_0\|_2 \to 0} \frac{\|r(x)\|_2}{\|x - x_0\|_2} = 0
\]
Denoting \( y = x - x_0 \), near \( x_0 \), we can approximate the nonlinear system by its linearization
\[
\dot{y} = \frac{\partial f}{\partial x}(x_0)y.
\]
The linearization of a nonlinear system tells us how the system behaves around \( x_0 \) in some cases. For instance,

**Theorem 2.1** If the Jacobian matrix \( \frac{\partial f}{\partial x}(x_0) \) is non-singular, then \( x_0 \) is an isolated equilibrium of the nonlinear system.

**Remark 2.2** The condition is sufficient, but not necessary. Think for instance of the system \( \dot{x} = x^3 \).

Another situation when the linearization might be useful is when deciding the stability properties of an equilibrium point. We will discuss this at depth in the chapter on Lyapunov stability – it is intrinsically related to the following (famous) result.

**Theorem 2.3 (Hartman-Grobman)** If the Jacobian matrix \( \frac{\partial f}{\partial x}(x_0) \) has no eigenvalues in the imaginary axis, then there exists \( \delta > 0 \) and a continuous map \( h : B(x_0, \delta) \to \mathbb{R}^n \), with a continuous inverse (i.e., a homeomorphism) mapping the trajectories of the nonlinear system onto the trajectories of the linearization.
3 Some classical and cool nonlinear examples

3.1 Bowing of a violin string

The following example is the mechanical analog of the bowing of a violin string (see Figure 1(a)). Bowing is modeled by the friction between the body and the conveyor belt. The conveyor belt is moving at a constant velocity $b$. Consider the second-order system

$$M \ddot{x} + kx + F_b(\dot{x}) = 0,$$

where the function $F_b$ is of the form plotted in Figure 1(b). Defining $x_1 = x$ and $x_2 = \dot{x}$, we can rewrite it as a system of first-order differential equations

$$\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{1}{M}(-kx_1 - F_b(x_2))
\end{align*}$$

The equilibrium points of the system are determined by

$$x_2 = 0, \quad -kx_1 - F_b(x_2) = 0$$

Therefore, there is only one equilibrium point, $x_0 = \left(-\frac{F_b(0)}{k}, 0\right)$, which is obviously isolated. To assess its stability properties, we linearize the system around it. The Jacobian at $x_0$ is

$$\frac{\partial f}{\partial x}(x_0) = \begin{bmatrix}
0 & 1 \\
\frac{1}{M} & -\frac{F_b'(0)}{M}
\end{bmatrix}$$

For instance, for $k = 3$, $M = 3$ and $F_b$ defined by

$$F_b(z) = \begin{cases}
-(z-b+c)^2 - d, & x < b \\
((z-b)c)^2 - d, & x > b
\end{cases}$$

with $b = 1$, $c = 2$ and $d = 3$, we get $x_0 = (\frac{4}{3}, 0)$ and

$$\frac{\partial f}{\partial x}(x_0) = \begin{bmatrix}
0 & 1 \\
-1 & \frac{1}{3}
\end{bmatrix}$$

which has eigenvalues $\frac{1}{3} \pm \frac{2\sqrt{2}}{3}i$. 

Figure 1: Bowing of a violin string.
3.2 The buckling beam

Consider the following model

\[ m \ddot{x} + d \dot{x} - \mu x + \lambda x + x^3 = 0 \]

Here \( x \) represents the one-dimensional deflection of the beam normal to the axial direction, \( \mu x \) stands for the applied axial force, \( \lambda x + x^3 \) models the restoring spring force in the beam, and \( d \dot{x} \) is the damping. We can rewrite the system as

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{1}{m} \left( dx_2 - \mu x_1 + \lambda x_1 + x_1^3 \right)
\end{align*}
\]

The equilibrium points are determined by

\[
x_2 = 0, \quad dx_2 - \mu x_1 + \lambda x_1 + x_1^3 = 0
\]

Therefore, for \( \lambda \leq \mu \), the system has three equilibrium points \((0,0)\), \((\pm \sqrt{\mu - \lambda}, 0)\). For \( \lambda > \mu \), the only equilibrium point is \((0,0)\). Figure 2 shows a phase portrait for the undamped case. The linearization of the system is

\[
\frac{\partial f}{\partial x}(x) = \begin{bmatrix} 0 & 1 \\ \frac{1}{m} (\mu - \lambda - 3x_1^2) & -\frac{d}{m} \end{bmatrix}
\]

For the values \( d = 1 \), \( m = 1 \), \( \mu = 2 \) and \( \lambda = 1 \), we get the equilibrium points \((0,0)\), \((-1,0)\), and \((1,0)\). The corresponding eigenvalues of the linearization at \((0,0)\) are

\[
-1 \pm \sqrt{5} \frac{2}{2}
\]

Figure 2: Phase portrait of the buckling beam for \( d = 1 \), \( m = 1 \), \( \mu = 2 \) and \( \lambda = 1 \).
and at \((\pm 1, 0)\),

\[
\frac{1}{2} \pm \frac{\sqrt{7}}{2}i
\]

Therefore, \((0, 0)\) is unstable and the other two equilibria are stable.

### 3.3 The rattleback

The rattleback (also known by the names of wobblestone and celtic stone) is a mechanical system with a rather curious behavior. The rattleback is a convex asymmetric rigid body rolling without sliding on a horizontal plane. It is known for its ability to spin in one direction and to resist spinning in the opposite direction for some values of the parameters (mass, moments of inertia), and for other values to exhibit multiple reversals, see Figure 3. A truly nonlinear example! Search for a video on youtube showing this amazing behavior.

The configuration of the rattleback is described by the coordinates \((x, y)\) in \(\mathbb{R}^2\) of the center of mass, and a rotation matrix \(R \in SO(3)\) specifying its orientation in space. The equations governing its motion are pretty complicated, so instead of writing them down, I’ll make a demo in class of its behavior. Hopefully, by the end of the course, this system will be a piece of cake to us.

![Figure 3: The rattleback (illustration taken from [3]).](image)

### 4 Planar dynamical systems

A planar dynamical system is represented by two scalar differential equations

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{align*}
\]  

(2)

The \((x_1, x_2)\)-plane is usually called the *phase plane*. The ‘bowing of a violin string’ and the ‘buckling beam’ above are both planar examples.

If \(t \mapsto x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2\) denotes the solution of the system that starts at \(x_0 \in \mathbb{R}^2\), the velocity vector of this curve in the \((x_1, x_2)\)-plane at time \(t\) is precisely the vector

\[
(\dot{x}_1(t), \dot{x}_2(t)) = f(x_1(t), x_2(t)) = (f_1(x_1(t), x_2(t)), f_2(x_1(t), x_2(t))).
\]

The vector field \(x \in \mathbb{R}^2 \rightarrow f(x) \in \mathbb{R}^2\) can be plotted with Mathematica/Matlab/Maple to give us an idea of the behavior of the solutions of the system. In Mathematica, use the command `PlotVectorField`. In Matlab, use the routine `pplane` provided by Prof. Polking at Rice University at [http://math.rice.edu/~dfield](http://math.rice.edu/~dfield).
Example 4.1 (Tunnel-diode circuit) Consider the following equations describing the behavior of a tunnel-diode circuit

\[
\begin{align*}
\dot{x}_1 &= .5(-h(x_1) + x_2) \\
\dot{x}_2 &= .2(-x_1 - 1.5x_2 + 1.2)
\end{align*}
\]

with \( h(x) = 17.76x - 103.79x^2 + 229.62x^3 - 226.31x^4 + 83.72x^5 \). Solving for the equilibrium points, we get

\[ Q_1 = (.063, .758), \quad Q_2 = (.285, .61), \quad Q_3 = (.884, .21). \]

Figure 4 shows the phase portrait of this system.

4.1 Linear systems on the plane

Consider the linear time-invariant system

\[
\dot{x} = Ax \tag{3}
\]

where \( A \) is a \( 2 \times 2 \) matrix. From linear algebra, we know that there exists an invertible matrix \( Q \) such that \( Q^{-1}AQ = J \), where \( J \) may take one of the following forms

\[
\begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix}, \quad \begin{bmatrix}
\lambda & 0 \\
0 & \lambda
\end{bmatrix}, \quad \begin{bmatrix}
\lambda & 1 \\
0 & \lambda
\end{bmatrix}, \quad \begin{bmatrix}
\alpha & -\beta \\
\beta & \alpha
\end{bmatrix}.
\]

The solution of (3) starting from \( x_0 \) is given by \( x(t) = Q \exp(Jt)Q^{-1}x_0 \). The first case (4) corresponds to the case when the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are real and distinct. The second and third case corresponds to the case when the eigenvalues are real and equal (second case, two linearly independent eigenvectors, third case only one). Finally, the fourth case corresponds to the case of complex eigenvalues \( \alpha \pm i\beta \).

(Un)stable (improper) node, saddle point, (un)stable focus, center: follow discussion in [2, Chapter 2.1]
4.2 Linearization of nonlinear systems around an equilibrium point

Consider the linearization of the nonlinear system (2) around an equilibrium point \( x_0 \in \mathbb{R}^2 \),

\[
\dot{y} = Ay
\]

where \( y \in \mathbb{R}^2 \) and

\[
A = \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix} = \frac{\partial f}{\partial x}(x_0).
\]

Recall the Hartman-Grobman theorem, cf. Theorem 2.3. From our previous discussion on linear systems, we deduce

(i) if the origin of the linearization is a stable (resp. unstable) node with distinct eigenvalues, then the equilibrium of the nonlinear system is a stable (respectively) unstable node;

(ii) if the origin of the linearization is a stable (resp. unstable) focus, then the equilibrium of the nonlinear system is a stable (respectively) focus;

(iii) if the origin of the linearization is a saddle point, then the equilibrium of the nonlinear system is a saddle point.

Example 4.2 (Tunnel-diode circuit revisited) Consider again the tunnel-diode circuit example. The Jacobian matrix of the system is

\[
\frac{\partial f}{\partial x}(Q_1) = \begin{bmatrix}
-3.598 & .5 \\
-2 & -3
\end{bmatrix}, \text{ eigenvalues } -3.57, -3.33 \Rightarrow \text{stable node}
\]

\[
\frac{\partial f}{\partial x}(Q_2) = \begin{bmatrix}
1.82 & .5 \\
-2 & -3
\end{bmatrix}, \text{ eigenvalues } 1.77, -0.25 \Rightarrow \text{saddle point}
\]

\[
\frac{\partial f}{\partial x}(Q_3) = \begin{bmatrix}
-1.427 & .5 \\
-2 & -3
\end{bmatrix}, \text{ eigenvalues } -1.33, -0.4 \Rightarrow \text{stable node}
\]

Figure 5 shows various solutions of this system.

If the origin of the linearization is a center point, then this is inconclusive for the nonlinear system.

Example 4.3 Consider the dynamical system

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - \varepsilon x_1^2 x_2
\end{align*}
\]

The linearization around the equilibrium point \((0,0)\) is

\[
\begin{align*}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= -y_1
\end{align*}
\]
4.2 Linearization of nonlinear systems around an equilibrium point

PLANAR DYNAMICAL SYSTEMS

\[
x' = 0.5(-17.76x + 103.79x^2 - 229.62x^3 + 226.31x^4 - 83.72x^5 + y)
\]
\[
y' = 0.2(-x - 1.5y + 1.2)
\]

Figure 5: Phase portrait of the tunnel-diode circuit example.

with eigenvalues \( \pm i \). Therefore, it is inconclusive about the stability properties of the equilibrium point of the nonlinear system (we cannot apply the Hartman-Grobman theorem). In fact, checking out the evolution of the function \( V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) \) along the solutions of the system, we get

\[
\frac{d}{dt}(V(x_1(t), x_2(t))) = -\varepsilon x_1^2 x_2^2.
\]

Therefore, for \( \varepsilon > 0 \), the equilibrium is stable, and for \( \varepsilon < 0 \) it is unstable (see Figure 6). Note how the linearization around the equilibrium is completely blind to this fact, i.e., no matter the value of \( \varepsilon \), it always gives the same answer (“the equilibrium of the linear system is a center”). We will learn how to invoke this kind of arguments systematically when we discuss Lyapunov stability analysis.

\[ x' = y, \quad y' = -x - \varepsilon x^2 y \]
\[ \varepsilon = 1 \]
\[ \varepsilon = -1 \]

Figure 6: Phase portraits for \( \varepsilon = 1 \) and \( \varepsilon = -1 \).

Example 4.4 The dynamical systems

\[
\dot{x} = x^3, \quad \dot{x} = -x^3
\]

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have the same linearization around 0, simply \( \dot{y} = 0 \). However, their behavior is very different!

5 Closed orbits and limit cycles

A non-trivial periodic solution of (2) satisfies \( x(t + T) = x(t) \) for all \( t \geq 0 \) and some period \( T > 0 \). Non-trivial excludes the possibility of constant solutions corresponding to equilibrium points. A closed orbit \( \gamma \) is the trace of the trajectory of a nontrivial periodic solution. An isolated closed orbit is called a limit cycle. All limit cycles are closed orbits, but not all closed orbits are limit cycles.

Example 5.1 (Duffing’s equation) Consider the following dynamical system

\[
\ddot{x} + \delta \dot{x} - x + x^3 = 0
\]

Writting this as a system of first-order differential equations, we get

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_1 - x_1^3 - \delta x_2
\end{align*}
\]

Figure 7(a) shows the phase portrait for \( \delta = 0 \). Note the infinitely many number of closed orbits.

5.1 Bendixson’s theorem

Theorem 5.2 (Bendixson’s theorem) Let \( D \subset \mathbb{R}^2 \) be a simply connected region. Consider the vector field \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) and assume that its divergence

\[
\text{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}
\]

is not identically zero and does not change sign on \( D \). Then the dynamical system \( \dot{x} = f(x) \) does not possess any closed orbits.
5.2 Poincaré-Bendixson theorem

Proof: Let us reason by contradiction. Assume there exists a closed orbit \( \gamma \). Along any orbit of \( f \), we have \( \frac{dx_2}{dx_1} = f_2/f_1 \). Therefore,

\[
\int_{\gamma} f_2(x_1, x_2)dx_1 - f_1(x_1, x_2)dx_2 = 0
\]

By Green’s theorem, this implies that

\[
\int \int_{S} \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx_1 dx_2 = 0
\]

where \( S \) is the interior of \( \gamma \). This contradict the hypothesis that \( \text{div}(f) \) is not identically zero and does not change sign on \( D \), and this concludes the proof.

Example 5.3 (Duffing’s equation revisited) Writing Duffing’s equation (cf. Example 5.1) as a system of first-order differential equations, we get

\[
\dot{x}_1 = x_2, \\
\dot{x}_2 = x_1 - x_3^3 - \delta x_2,
\]

and recognize \( f(x_1, x_2) = (x_2, x_1 - x_3^3 - \delta x_2) \). We compute \( \text{div}(f) = -\delta \). So for \( \delta \neq 0 \), there are no closed orbits on \( \mathbb{R}^2 \)!

Figure 7(b) shows the phase portrait for \( \delta = 1 \). Note that the closed orbits of Figure 7(a) have all disappeared.

5.2 Poincaré-Bendixson theorem

A set \( M \) is called positively invariant for the dynamical system (2) if every trajectory starting in \( M \) stays in \( M \) for all future time.

Theorem 5.4 (Poincaré-Bendixson theorem) Let \( M \subset \mathbb{R}^2 \) be a non-empty, closed, bounded, positively invariant set. Then \( M \) contains an equilibrium point or a closed orbit.

The intuition behind this result is the trajectories in the set \( M \) will have to approach closed orbits or equilibrium points as time tends to infinity.

Remark 5.5 (How to find invariant sets?) Consider a simple closed curve defined by the equation \( V(x) = c \), where \( V : \mathbb{R}^n \to \mathbb{R} \) is continuously differentiable. We know that the gradient \( \nabla V \) of \( V \) points in the direction of maximum increase of the function. Actually, all directions whose inner product with \( \nabla V \) is positive are directions in which \( V \) increases. Therefore, we look at the inner product

\[
\nabla V \cdot f = \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2
\]

If this is less than or equal to zero at each point in the curve defined by \( V(x) = c \), that means that the vector field \( f \) is pointing inward or is tangent to the curve, and therefore, the trajectories of the dynamical system are trapped inside the closed region defined by the curve.

Example 5.6 (Harmonic oscillator) Consider the dynamical system

\[
\dot{x}_1 = x_2 \\
\dot{x}_2 = -x_1
\]
and the annular region \( M = \{ x \in \mathbb{R}^2 \mid c_1 \leq x_1^2 + x_2^2 \leq c_2 \} \), with \( c_2 > c_1 > 0 \). The set \( M \) is closed, bounded, and does not contain any equilibrium point. Moreover, since \( f(x) \cdot \nabla V(x) = 0 \) everywhere, trajectories are trapped inside \( M \), i.e., the set is positively invariant. Hence, by Poincaré-Bendixson theorem, we know that there is at least a periodic orbit in \( M \). Actually, there are plenty!

**Example 5.7 (Lotka-Volterra predator-prey model with limited growth)** Motivated by the study of the cyclic variation of the populations of small fish in the Adriatic, Count Vito Volterra proposed the following model for the population evolution of two species: prey \( x \) and predators \( y \),

\[
\begin{align*}
\dot{x} &= (a - by - \lambda x)x, \\
\dot{y} &= (cx - d - \mu y)y,
\end{align*}
\]

where \( a, b, c, d, \lambda, \mu > 0 \). The equilibria of the system are

\[
\begin{align*}
e_1 &= (0, 0), & e_2 &= \left( \frac{a}{\lambda}, 0 \right), & e_3 &= \left( \frac{bd + a\mu}{bc + \lambda\mu}, \frac{ac - d\lambda}{bc + \lambda\mu} \right), & e_4 &= (0, -\frac{d}{\mu})
\end{align*}
\]

We neglect the last one, since our attention is restricted to positive values \( x, y \geq 0 \) for the populations. For the same reason, the third one is not considered if \( ac - d\lambda < 0 \). The Jacobian of the vector field takes the form

\[
\frac{\partial f}{\partial x} = \begin{bmatrix} a - by - 2\lambda x & -bx \\ cy & cx - d - 2\mu y \end{bmatrix}
\]

From here, we get

\[
\begin{align*}
\frac{\partial f}{\partial x}(e_1) &= \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}, & \text{eigenvalues are } a, -d \Rightarrow \text{saddle point} \\
\frac{\partial f}{\partial x}(e_2) &= \begin{bmatrix} a & \frac{b}{x} \\ 0 & \frac{c}{x} - d \end{bmatrix}, & \text{eigenvalues are } -a, \frac{ac}{\lambda} - d \Rightarrow \text{saddle point or stable node} \\
\frac{\partial f}{\partial x}(e_3) &= \ldots
\end{align*}
\]

Consider the following two cases

**Case** \( ac < d\lambda \): In this case, we only have the equilibria \( e_1 \) and \( e_2 \). The first one is a saddle point and the second one is a stable node. A sample phase portrait is plotted in Figure 8. Note that the region \( A \cup B \) is non-empty, closed, bounded and positively invariant. By Poincaré-Bendixson, this region must contain an equilibrium point or a closed orbit. Note that \( \dot{y} \leq 0 \) in this region, therefore there are no closed orbits. Also, the trajectories starting in \( C \) eventually reach \( B \) (why?). Therefore, all trajectories eventually converge to either \( e_1 \) or \( e_2 \).

**Case** \( ac > d\lambda \): In this case, we have the equilibria \( e_1, e_2 \) and \( e_3 \). The first two ones are saddle points and the third one is a stable focus/node (check!). A sample phase portrait is plotted in Figure 9. The region plotted in the figure is non-empty, closed, bounded and positively invariant. By Poincaré-Bendixson, it contains an equilibrium point or a closed orbit. By index theory, if such an orbit exists, it must enclose the equilibrium \( e_3 \). All trajectories starting outside this region eventually enter it. Therefore, all trajectories will eventually converge to the equilibria \( e_1, e_2, e_3 \), or the limit cycle if it exists.

This example shows how, without actually computing the explicit solutions of a dynamical system, we can deduce a lot of things about its behavior with the qualitative tools that we have introduced. •
6 Bifurcations

Systems of physical interest have typically parameters which appear in the equations governing their evolution. Here we restrict our attention to the planar case, \( n = 2 \). Consider then
\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, \mu) \\
\dot{x}_2 &= f_2(x_1, x_2, \mu)
\end{align*}
\] (5)

where \( \mu \in \mathbb{R} \). We will also use the short-hand notation \( \dot{x} = f_\mu(x) \).

One issue of practical importance is whether the system maintains its qualitative behavior (i.e., the pattern of its equilibrium points and periodic orbits, as well as their stability properties) under different values of the parameters. When the system is robust to infinitesimally small perturbations then it is called structurally stable. Instead, here we are interested in the complementary situation. In particular, we are interested in perturbations that will change the equilibrium points or periodic orbits of the system or change their stability properties.

**Example 6.1** Consider the system
\[
\begin{align*}
\dot{x}_1 &= \mu - x_1^2 \\
\dot{x}_2 &= -x_2
\end{align*}
\]

For \( \mu > 0 \), there are two equilibrium points \( e_1 = (\sqrt{\mu}, 0) \) and \( e_2 = (-\sqrt{\mu}, 0) \). The linearizations are, respectively,
\[
\begin{bmatrix}
-2\sqrt{\mu} & 0 \\
0 & -1
\end{bmatrix}, \quad \begin{bmatrix}
2\sqrt{\mu} & 0 \\
0 & -1
\end{bmatrix}.
\]

Therefore, \( e_1 \) is a stable node and \( e_2 \) is a saddle point. As \( \mu \) decreases to zero, both equilibrium points approach, eventually collide and disappear. Figure 10 shows various phase portraits. The observed change in the qualitative behavior of the system is called a bifurcation.
A bifurcation is a change in the equilibrium points or periodic orbits, or in their stability properties, as a parameter is varied. The parameter is called a bifurcation parameter, and the values at which changes occur are called bifurcation points. In the previous example, the bifurcation parameter is $\mu$ and the bifurcation point is 0.

**Example 6.2 (Jet engine compressor)** The following is a practical example of the importance of bifurcations. Consider the following system of equations

$$
\dot{x}_1 = B(C(x_1) - x_2),
\dot{x}_2 = \frac{1}{B}(x_1 - F_\alpha^{-1}(x_2)),
$$

where $B > 0$ is the non-dimensional compressor speed, $\alpha$ is the throttle area, $x_1$ is the compressor mass flow and $x_2$ is the plenum pressure rise in the compressor unit. The function $C$ corresponds to the compressor characteristic and the function $F_\alpha$ to the throttle characteristic. For definiteness, we assume that $C$ and $F$
have the following form

\[ C(x) = -x^3 + \frac{3}{2}(b + a)x^2 - 3abx + \frac{1}{2}(2c + 3ab^2 - b^3), \]

\[ F_\alpha(x) = \frac{x^2}{\alpha^2} \text{sign}(x), \]

where \( \text{sign}(x) = 1 \) if \( x > 0 \), \( \text{sign}(x) = 0 \) if \( x = 0 \) and \( \text{sign}(x) = -1 \) if \( x < 0 \). Here, \( a, b, c \) are positive constants determining the qualitative shape of the compressor characteristic. Figure 11 shows phase portraits of this system for different values of \( \alpha \).

The equilibrium points of this system are determined by the equation

\[ x_2 = C(x_1) = F_\alpha(x_1), \]

i.e., they occur at the points of intersection of the compressor and throttle characteristic (depending on the shape of these functions, we might have multiple equilibria).

For \( \alpha = 1 \), there is a single equilibrium. The Jacobian of the vector field is

\[ Df(x) = \begin{bmatrix} BC'(x_1) & -B \\ \frac{1}{\alpha} & G'_\alpha(x_2) \end{bmatrix} \]

where \( G_\alpha = F_\alpha^{-1} \). For sufficiently small compressor speed \( B \), this equilibrium is stable (check!), see Figure 12 (left). When \( B \) is increased, this equilibrium becomes unstable, see Figure 12 (center). Actually, can you identify a positively invariant set enclosing the unstable equilibrium and apply Poincaré-Bendixson’s theorem to conclude that there must exist the limit cycle observed in the plot? This limit cycle is stable. This bifurcation is called supercritical Hopf bifurcation. When the compressor speed \( B \) is larger, a strange phenomenon occurs to the limit cycle, see Figure 12 (right).

For equation (5), the equilibria are determined by

\[ f_\mu(x) = 0 \]

and therefore, they are a function of \( \mu \), i.e. \( x^\star(\mu) \). The implicit function theorem implies that \( x^\star(\mu) \) is a smooth function of \( \mu \) as long as the Jacobian linearization around \( x^\star(\mu) \), denoted

\[ D_x f_\mu(x^\star(\mu)) = \frac{\partial f_\mu}{\partial x}(x^\star(\mu)) \]

does not have a zero eigenvalue. What happens then when \( D_x f_\mu(x^\star(\mu)) \) has a zero eigenvalue? Several branches of equilibria come together and a bifurcation occurs.

Figure 11: From left to right, phase portrait of jet engine example for \( \alpha = .75, 1, 1.25 \).
Consider Example 6.1. In this case, we have the two functions $x^*_1(\mu) = (\sqrt{\mu}, 0)$ and $x^*_2(\mu) = (-\sqrt{\mu}, 0)$. The Jacobian linearization

$$D_x f_{\mu}(x^*_1(\mu)) = \begin{bmatrix} -2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix}$$

has a zero eigenvalue precisely when $\mu = 0$, which is when the two branches of equilibria collide.

### 6.1 Bifurcations of one-dimensional systems

#### 6.1.1 Supercritical/Subcritical pitchfork bifurcation

Consider the system

$$\dot{x} = \mu x - x^3$$

The equilibria are 0 and $\pm \sqrt{\mu}$. The bifurcation point is $\mu = 0$. For $\mu < 0$, the only equilibrium, 0, is stable. For $\mu > 0$, 0 is unstable and $\pm \sqrt{\mu}$ are stable.

#### 6.1.2 Transcritical bifurcation or exchange of stability

Consider the system

$$\dot{x} = \mu x - x^2$$

The equilibria are 0 and $\mu$. The bifurcation point is $\mu = 0$. For $\mu < 0$, 0 is stable and $\mu$ is unstable. For $\mu > 0$, 0 is unstable and $\pm \sqrt{\mu}$ are stable.

#### 6.1.3 Fold bifurcation

Consider the system

$$\dot{x} = \mu - x^2$$

The equilibria are $\pm \sqrt{\mu}$. The bifurcation point is $\mu = 0$. For $\mu < 0$, there are no equilibrium points. For $\mu > 0$, $\sqrt{\mu}$ is stable and $-\sqrt{\mu}$ is unstable.

Figure 12: From left to right, phase portrait of jet engine example with $\alpha = 1$, and $B = .1, 3, 1$. 

Example 6.3 Consider Example 6.1. In this case, we have the two functions $x^*_1(\mu) = (\sqrt{\mu}, 0)$ and $x^*_2(\mu) = (-\sqrt{\mu}, 0)$. The Jacobian linearization

$$D_x f_{\mu}(x^*_1(\mu)) = \begin{bmatrix} -2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix}$$

has a zero eigenvalue precisely when $\mu = 0$, which is when the two branches of equilibria collide.
6.2 Bifurcations of planar systems

6.2.1 Saddle-node bifurcation

Example 6.1.

6.2.2 Supercritical/Subcritical Hopf bifurcation

Consider the system

\[ \begin{align*}
\dot{x}_1 &= x_1(\mu - x_1^2 - x_2^2) - x_2 \\
\dot{x}_2 &= x_2(\mu - x_1^2 - x_2^2) + x_1
\end{align*} \]

In polar coordinates, the system looks like

\[ \begin{align*}
\dot{r} &= \mu r - r^3 \\
\dot{\theta} &= 1
\end{align*} \]

The only equilibrium is the point (0, 0). The Jacobian at the origin is

\[ \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix} \]

with eigenvalues \( \mu \pm i \). For \( \mu < 0 \), the equilibrium is a stable focus, and all trajectories are attracted to it. Instead, for \( \mu > 0 \), the equilibrium is an unstable focus, but there is a limit cycle that attracts all solutions except for the equilibrium. The limit cycle has radius \( r = \sqrt{\mu} \). This is the supercritical Hopf bifurcation (cf. Figure 13).

![Figure 13: Supercritical Hopf bifurcation (left, \( \mu < 0 \), right, \( \mu > 0 \)).](image)

The subcritical Hopf bifurcation is exemplified by the system

\[ \begin{align*}
\dot{x}_1 &= x_1(\mu + x_1^2 + x_2^2 - (x_1^2 + x_2^2)^2) - x_2 \\
\dot{x}_2 &= x_2(\mu + x_1^2 + x_2^2 - (x_1^2 + x_2^2)^2) + x_1
\end{align*} \]
6.2 Bifurcations of planar systems

which in polar coordinates reads

\[ \dot{r} = \mu r + r^3 - r^5 \]
\[ \dot{\theta} = 1 \]

The only equilibrium is the point \((0, 0)\). For \( \mu < 0 \), the equilibrium is a stable focus. Instead, for \( \mu > 0 \), the equilibrium is an unstable focus. From the equation

\[ \mu + r^2 - r^4 = 0 \]

we can determine the limit cycles. The solutions to this equation are

\[ r^2 = \frac{1}{2} \left( 1 \pm \sqrt{1 + 4\mu} \right) \]

For \( \mu < 0 \), there are two limit cycles. By looking at the equation \( \dot{r} = \mu r + r^3 - r^5 \), one can see that the limit cycle at \( r^2 = \frac{1}{2} (1 + \sqrt{1 + 4\mu}) \) is stable, whereas the one at \( r^2 = \frac{1}{2} (1 - \sqrt{1 + 4\mu}) \) is unstable. For small \( |\mu| \), the unstable limit cycle can be approximated by \( r^2 = -\mu \).

For \( \mu > 0 \), there is only one limit cycle at \( r^2 = \frac{1}{2} (1 + \sqrt{1 + 4\mu}) \), which is stable. Figure 14 illustrates this example.

6.2.3 One example of a global bifurcation: homoclinic bifurcation

The previous bifurcations occur in the vicinity of equilibrium points, i.e., they are local. There are also global bifurcations, which involve large regions of the state plane. Here we only give one example. Other examples can be found in [4]. Consider the system

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = \mu x_2 + x_1 - x_1^2 + x_1 x_2 \]

There are two equilibrium points, \( e_1 = (0, 0) \) and \( e_2 = (1, 0) \). Linearizing, one can see that \( e_1 \) is always a saddle, while \( e_2 \) is an unstable focus for \(-1 < \mu < 1\).
The bifurcation point is $\mu_c \approx -0.8645$. For $\mu < \mu_c$, there is a stable limit cycle that encloses the unstable focus. As $\mu$ approaches $\mu_c$, the trajectory swells and finally touches the saddle at $\mu = \mu_c$, creating a trajectory that starts and ends at the saddle called homoclinic orbit. Note that this trajectory is not a limit cycle anymore because it takes an infinite time to traverse it. For $\mu > \mu_c$, the limit cycle disappears. The bifurcation occurs without any changes to the equilibrium points. This bifurcation, illustrated in Figure 15 is called saddle-connection or homoclinic bifurcation.

![Figure 15: Homoclinic bifurcation.](image)

References

