Nonlinear Control - MAE281b

Final

Student name and number ____________________________

Please be accurate in the presentation of your solutions, and quote the results from class that you are using.

1. (3 points) Consider the following model for a unicycle that can only move forward and rotate in place:

\[
\begin{align*}
\dot{x} &= u^2 \cos \theta \\
\dot{y} &= u^2 \sin \theta \\
\dot{\theta} &= v
\end{align*}
\]

Given a desired arbitrary configuration \((x_*, y_*, \theta_*)\), can you design a continuous feedback stabilizer that makes \((x_*, y_*, \theta_*)\) an asymptotically stable point of the closed-loop system? If you can, design it using any of the methods seen in class. If you cannot, justify your answer.

2. (3 points) Consider a unicycle subject to some drift (let us assume that the wind is blowing, pushing it in some direction),

\[
\begin{align*}
\dot{x} &= u \cos \theta \\
\dot{y} &= -y + u \sin \theta \\
\dot{\theta} &= v
\end{align*}
\]

Find a control-Lyapunov function relative to \((0, 0, 0)\) that satisfies the small-control property. Use this fact to design a continuous local feedback stabilizer that makes \((0, 0, 0)\) an asymptotically stable point of the closed-loop system.

3. (3 points) Consider a planar pendulum which can spin about a fixed vertical axis at some rate \(\omega \geq 0\). The equations describing the pendulum motion are

\[
\begin{align*}
\ddot{\theta} + b \dot{\theta} + \sin \theta (\omega_c^2 - \omega^2 \cos \theta) &= 0, \\
\dot{\omega} &= u,
\end{align*}
\]

where \(b \geq 0\) and \(\omega_c > 0\) are constant parameters. The second equation means that we can control the spinning velocity of the pendulum. Further assume that we can measure the angular position of the pendulum

\[y = \theta.\]

Does the system (1) has a relative degree? Where? Use your answer to stabilize the system around \((\pi/4, 0, c)\), for some desired angular velocity \(c > 0\).
4. (3 points) Consider the system
\[
\dot{x}_1 = u \\
\dot{x}_2 = x_1 \\
\dot{x}_3 = x_1^3 + x_2^2
\]
Is it small-time locally accessible from 0? Is it small-time locally controllable from 0? Carefully justify your answers.

5. (3 points) Consider the system
\[
\dot{x}_1 = -x_1 + x_1^2 x_2 \\
\dot{x}_2 = u \\
y = x_2
\]
Do the following:

(i) Find a state feedback of the form \(u = \alpha(x) + \beta(x)v\) such that the resulting system with input \(v\) and output \(y\) is passive with storage function \(V(x) = \frac{1}{2}(x_1^2 + x_2^2)\)

(ii) Is the system with input \(v\) and output \(y\) zero-state observable?

(iii) Use (i) and (ii) to design a controller that globally stabilizes \((0, 0)\). Justify your answer.

6. (Extra with a catch: you have to answer it perfectly to get 3 extra points) Consider a control-affine system \(\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x)\). Assume that there exists a continuously differentiable function \(V : \mathbb{R}^n \to \mathbb{R}\) which is positive definite, globally proper, and satisfies
\[
\mathcal{L}_f V(x) \leq 0 \quad \text{for all } x \in \mathbb{R}^n.
\]
Let
\[
Q = \{ x \in \mathbb{R}^n \mid \mathcal{L}_f V(x) = 0 \text{ and } \mathcal{L}_f^k \mathcal{L}_g_i V(x) = 0, k = 0, 1, 2, \ldots, i = 1, \ldots, m \},
\]
and assume that \(Q = 0\). Prove that the feedback law
\[
k(x) = -(\mathcal{L}_{g_1} V(x), \ldots, \mathcal{L}_{g_m} V(x))
\]
globally asymptotically stabilizes the system.

Hint: Be mindful of what you need to prove. Stability+attractivity+global! To show the attractivity part, you can use the LaSalle Invariance Principle. If somebody says again that \(\mathbb{R}^d\) is a compact set, (s)he gets an F!