Distributed Control of Robotic Networks

A Mathematical Approach to Motion Coordination Algorithms

Chapter 4: Connectivity maintenance and rendezvous

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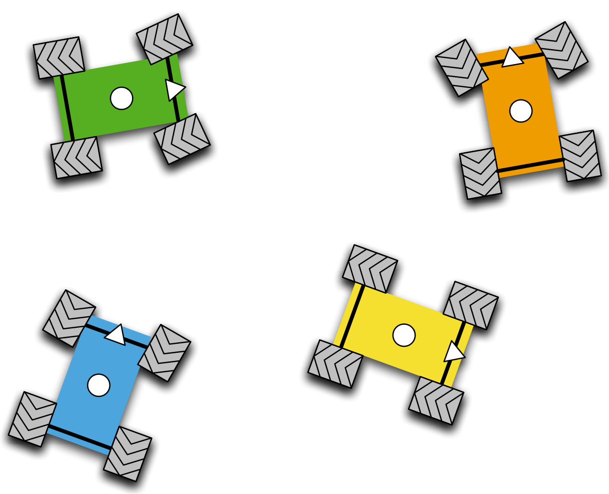

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Chapter Four

Connectivity maintenance and rendezvous

The aims of this chapter are twofold. First, we introduce the rendezvous problem and analyze various coordination algorithms that achieve it, providing upper and lower bounds on their time complexity. Second, we introduce the problem of maintaining connectivity among a group of mobile robots and use geometric approaches to preserve this topological property of the network.

Loosely speaking, the *rendezvous objective* is to achieve agreement over the physical location of as many robots as possible, that is, to steer the robots to a common location. This objective is to be achieved with the limited information flow described in the model of the network. Typically, it will be impossible to solve the rendezvous problem for all robots if the robots are placed in such a way that they do not form a connected communication graph. Therefore, it is reasonable to assume that the network is connected at initial time, and that a good property of any rendezvous algorithm is that of maintaining some form of connectivity among robots. This discussion motivates the *connectivity maintenance problem*. Once a model for when two robots can acquire each other's relative position is adopted, this problem is of particular relevance, as the inter-robot topology depends on the physical states of the robots. Our exposition here is mainly based on Ando et al. (1999), Cortés et al. (2006), and Ganguli et al. (2009).

The chapter is organized as follows. In the first section, we formally introduce the two coordination problems. In the second section, we define various connectivity constraint sets to limit the motion of robots in order to maintain network connectivity. These notions of constraint sets allows us to study in the next section various rendezvous algorithms with connectivity maintenance properties. We study numerous variations of the circumcenter algorithm for the rendezvous objective and we characterize its complexity. Additionally, we introduce the perimeter-minimizing algorithm for nonconvex environments. The fourth section presents various simulations of the proposed motion-coordination algorithms. We end the chapter with three sections on, respectively, bibliographic notes, proofs of the results presented in the chapter, and exercises. Our technical treatment is based on the LaSalle Invariance Principle, on linear distributed algorithms, and on geometric tools such as proximity graphs and robust visibility.

4.1 PROBLEM STATEMENT

We begin this section by reviewing the classes of networks and the types of problems that will be considered in the chapter.

4.1.1 Networks with discrete-time motion

In the course of the chapter, we will consider the robotic networks $\mathcal{S}_{\text{disk}}$, $\mathcal{S}_{\text{LD}}$, and $\mathcal{S}_{\infty\text{-disk}}$, and the relative-sensing networks $\mathcal{S}_{\text{disk}}^{\text{rs}}$ and $\mathcal{S}_{\text{vis-disk}}^{\text{rs}}$ presented in Example 3.4 and in Section 3.2.2.

For the robotic networks $\mathcal{S}_{\text{disk}}$, $\mathcal{S}_{\text{LD}}$, and $\mathcal{S}_{\infty\text{-disk}}$, we will, however, assume that the robots move in discrete time, that is, we adopt the discrete-time motion model

$$p^{[i]}(\ell+1) = p^{[i]}(\ell) + u^{[i]}(\ell), \quad i \in \{1, \dots, n\}. \tag{4.1.1}$$

Similarly, for the relative-sensing networks $\mathcal{S}_{\text{disk}}^{\text{rs}}$ and $\mathcal{S}_{\text{vis-disk}}^{\text{rs}}$, we adopt the discrete-time motion model

$$p_{\text{fixed}}^{[i]}(\ell+1) = p_{\text{fixed}}^{[i]}(\ell) + R_{\text{fixed}}^{[i]} u_i^{[i]}(\ell), \quad i \in \{1, \dots, n\}. \tag{4.1.2}$$

As an aside, if we express the previous equation with respect to frame i at time t, then equation (4.1.2) reads

$$p_{(\text{frame } i \text{ at time } \ell)}^{[i]}(\ell+1) = u_{(\text{frame } i \text{ at time } \ell)}^{[i]}(\ell), \quad i \in \{1, \dots, n\}.$$

We present the treatment in discrete time for simplicity. It is easy to show that any control law for the discrete-time motion model can be implemented in the continuous-time networks. In what follows, we begin our discussion by assuming no bound on the control magnitude and we later introduce an upper bound denoted by $u_{\max}$.

4.1.2 The rendezvous task

Next, we discuss the rendezvous problem. There are different ways of formulating this objective in terms of task maps. Let $\mathcal{S} = (\{1, \dots, n\}, \mathcal{R}, E_{\text{cmm}})$ be a uniform robotic network. The *(exact) rendezvous task* $\mathcal{T}_{\text{rndzvs}} : X^n \rightarrow$

$\{\texttt{true}, \texttt{false}\}$ for $\mathcal{S}$ is the coordination task defined by

$$\mathcal{T}_{\text{rndzvs}}(x^{[1]}, \dots, x^{[n]}) = \begin{cases} \texttt{true}, & \text{if } x^{[i]} = x^{[j]}, \text{ for all } (i,j) \in E_{\text{cmm}}(x^{[1]}, \dots, x^{[n]}), \\ \texttt{false}, & \text{otherwise.} \end{cases}$$

Next, assume that, for the same network $\mathcal{S} = (\{1, \dots, n\}, \mathcal{R}, E_{\text{cmm}})$, the robots' physical state space is $X \subset \mathbb{R}^d$. It is convenient to review some basic notation consistent with what we adopted in Chapter 2. We let $\mathcal{P} = \{p^{[1]}, \dots, p^{[n]}\}$ denote the set of agents' location in $X \subset \mathbb{R}^d$ and we let P be an array of n points in $\mathbb{R}^d$. Furthermore, we let avrg denote the average of a finite point set in $\mathbb{R}^d$, that is,

$$\text{avrg}(\{q_1, \dots, q_k\}) = \frac{1}{k}(q_1 + \cdots + q_k).$$

For $\varepsilon \in \mathbb{R}_{>0}$, the *$\varepsilon$-rendezvous task* $\mathcal{T}_{\varepsilon\text{-rndzvs}} : (\mathbb{R}^d)^n \to \{\texttt{true}, \texttt{false}\}$ for $\mathcal{S}$ is defined as follows: $\mathcal{T}_{\varepsilon\text{-rndzvs}}$ is $\texttt{true}$ at P if and only if each robot position $p^{[i]}$, for $i \in \{1, \dots, n\}$, is at distance less than ε from the average position of its E_{cmm}-neighbors. Formally,

$$\mathcal{T}_{\varepsilon\text{-rndzvs}}(P) = \texttt{true} \iff \|p^{[i]} - \text{avrg}\left(\{p^{[j]} \mid (i,j) \in E_{\text{cmm}}(P)\}\right)\|_2 < \varepsilon, \quad i \in \{1, \dots, n\}.$$

4.1.3 The connectivity maintenance problem

Assume that the communication graph, computed as a function of the robot positions, is connected: How should the robots move in such a way that their communication graph is again connected? Clearly, the problem depends upon: (1) how the robots move; and (2) what proximity graph describes the communication graph or, in the case of relative-sensing networks, what sensor model is available on each robot.

The key idea is to restrict the allowable motion of each agent. Different motion constraint sets correspond to different communication or sensing graphs. We have three objectives in doing so. First, we aim to achieve this objective only based on local measurements or 1-hop communication, that is, without introducing processor states explicitly dedicated to this task. Second, the constraint sets should depend continuously on the position of the robots. Third, we have the somehow informal objective to design the constraint sets as "large" as possible so as to minimally constrain the motion of the robots.

4.2 CONNECTIVITY MAINTENANCE ALGORITHMS

In this section, we present some algorithms that might be used by a robotic network to maintain communication connectivity. The results presented in this section start with the original idea introduced by Ando et al. (1999) for first-order robots communicating along the edges of a disk graph, that is, for the network described in Example 3.4. This idea is then generalized to a number of useful settings. The properties of proximity graphs presented in Section 2.2 play a key role in formulating and solving the connectivity problem.

4.2.1 Enforcing range-limited links

First, we aim to constrain the motion of two first-order agents in order to maintain a communication link between them. We assume that the communication takes place over the disk graph $\mathcal{G}_{\text{disk}}(r)$ with communication range $r > 0$.

Loosely stated, the *pairwise connectivity maintenance problem* is as follows: given two neighbors in the proximity graph $\mathcal{G}_{\text{disk}}(r)$, find a rich set of control inputs for both agents with the property that, after moving, both agents are again within distance r. We provide a solution to this problem as follows.

Definition 4.1 (Pairwise connectivity constraint set). Consider two agents i and j at positions $p^{[i]} \in \mathbb{R}^d$ and $p^{[j]} \in \mathbb{R}^d$ such that $\|p^{[i]} - p^{[j]}\|_2 \leq r$. The *connectivity constraint set* of agent i with respect to agent j is

$$\mathcal{X}_{\text{disk}}(p^{[i]}, p^{[j]}) = \overline{B}\Big(\frac{p^{[j]} + p^{[i]}}{2}, \frac{r}{2}\Big). \qquad \bullet$$

Note that both robots, i and j, can independently compute their respective connectivity constraint sets. The proof of the following result is straightforward.

Lemma 4.2 (Maintaining pairwise connectivity). *Assume that the distance between agents $p^{[i]}$ and $p^{[j]}$ is no more than r, at some time ℓ. If the control $u^{[i]}(\ell)$ takes value in*

$$u^{[i]}(\ell) \in \mathcal{X}_{\text{disk}}\big(p^{[i]}(\ell), p^{[j]}(\ell)\big) - p^{[i]}(\ell) = \overline{B}\Big(\frac{p^{[j]}(\ell) - p^{[i]}(\ell)}{2}, \frac{r}{2}\Big),$$

and, similarly, $u^{[j]}(\ell) \in \mathcal{X}_{\text{disk}}\big(p^{[j]}(\ell), p^{[i]}(\ell)\big) - p^{[j]}(\ell)$, then, according to the discrete-time motion model (4.1.1):

(i) the positions of both agents at time $\ell+1$ *are inside the connectivity constraint set* $\mathcal{X}_{\text{disk}}\big(p^{[i]}(\ell), p^{[j]}(\ell)\big)$*; and*

(ii) the distance between the agents' positions at time $\ell+1$ *is no more than* r*.*

We illustrate these pairwise connectivity maintenance concepts in Figure 4.1.

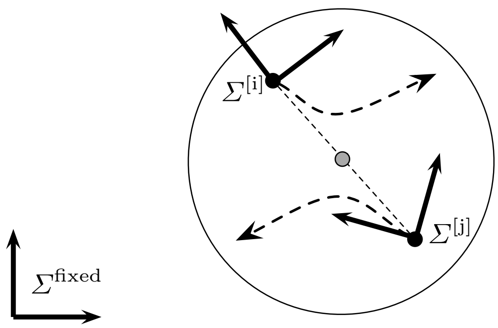


Figure 4.1 An illustration of the connectivity maintenance constraint. Starting from positions $p^{[i]}$ and $p^{[j]}$, the robots are restricted to moving inside the disk centered at $\mathcal{X}_{\text{disk}}(p^{[i]}, p^{[j]}) = \frac{1}{2}\big(p^{[i]} + p^{[j]}\big)$ with radius $\frac{r}{2}$.

Remark 4.3 (Constraints for relative-sensing networks). Let us consider a relative-sensing network with a disk sensor of radius r (see Example 3.15). Recall the following facts about this model. First, agent i measures the position of robot j in its frame $\Sigma^{[i]}$, that is, robot i measures $p_i^{[j]}$. Second, $p_i^{[i]} = \mathbf{0}_d$. Third, if $W \subset \mathbb{R}^d$, then W_i denotes its expression in the frame $\Sigma^{[i]}$. Combining these notions and assuming that the inter-agent distance is no more than r, the pairwise connectivity constraint set in Definition 4.1 satisfies

$$\Big(\mathcal{X}_{\text{disk}}(p^{[i]}, p^{[j]})\Big)_i = \mathcal{X}_{\text{disk}}\big(\mathbf{0}_d, p_i^{[j]}\big) = \overline{B}\Big(\frac{p_i^{[j]}}{2}, \frac{r}{2}\Big). \qquad \bullet$$

4.2.2 Enforcing network connectivity

Here, we focus on how to constrain the mobility of multiple agents in order to maintain connectivity for the entire network that they form. We again consider the case of first-order agents moving according to the discrete-time equation (4.1.1) and communicating over $\mathcal{G}_{\text{disk}}(r)$.

Loosely stated, the *network connectivity maintenance problem* is as follows: Given n agents at positions $\mathcal{P}(\ell) = \{p^{[1]}(\ell), \dots, p^{[n]}(\ell)\}$ in which they form a connected r-disk graph $\mathcal{G}_{\text{disk}}(r)$, the objective is to find a rich set of control inputs for all agents with the property that, at time $\ell + 1$, the agents' new positions $\mathcal{P}(\ell+1)$ again form a connected r-disk graph $\mathcal{G}_{\text{disk}}(r)$. We provide a simple, but potentially conservative, solution to this problem as follows.

Definition 4.4 (Connectivity constraint set for groups of agents). Consider a group of agents at positions $\mathcal{P} = \{p^{[1]}, \dots, p^{[n]}\} \subset \mathbb{R}^d$. The *connectivity constraint set* of agent i with respect to $\mathcal{P}$ is

$$\mathcal{X}_{\text{disk}}(p^{[i]}, \mathcal{P}) = \big\{x \in \mathcal{X}_{\text{disk}}(p^{[i]}, q) \mid q \in \mathcal{P} \setminus \{p^{[i]}\} \text{ s.t. } \|q - p^{[i]}\|_2 \leq r\big\}. \quad \bullet$$

In other words, if $q_1, \dots, q_l$ are agents' positions whose distance from $p^{[i]}$ is no more than r, then the connectivity constraint set for agent i is the intersection of the constraint sets $\overline{B}\big(\frac{1}{2}(q_k + p^{[i]}), \frac{r}{2}\big)$ for $k \in \{1, \dots, l\}$ (see Figure 4.2).

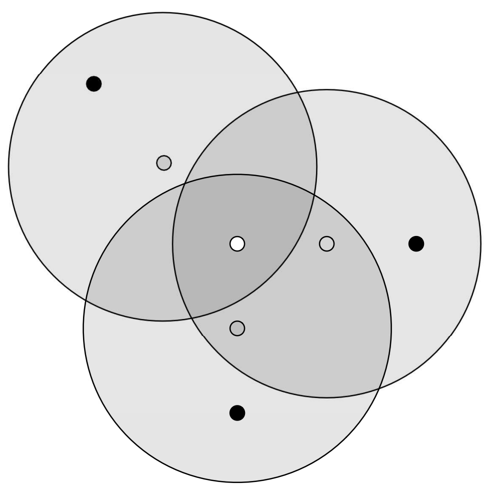


Figure 4.2 An illustration of network connectivity maintenance. The connectivity $\mathcal{X}_{\text{disk}}$-constraint set of the white-colored agent is the intersection of the individual constraint sets determined by its neighbors.

The following result is a consequence of Lemma 4.2.

Lemma 4.5 (Maintaining network connectivity). *Consider a group of agents at positions $\mathcal{P}(\ell) = \{p^{[1]}(\ell), \dots, p^{[n]}(\ell)\} \subset \mathbb{R}^d$ at time ℓ. If each* *agent's control* $u^{[i]}(\ell)$ *takes value in*

$$u^{[i]}(\ell) \in \mathcal{X}_{\text{disk}}\big(p^{[i]}(\ell), \mathcal{P}(\ell)\big) - p^{[i]}(\ell), \quad i \in \{1, \dots, n\},$$

then, according to the discrete-time motion model (4.1.1):

(i) each agent remains in its connectivity constraint set, that is, $p^{[i]}(\ell+1) \in \mathcal{X}_{\text{disk}}(p^{[i]}(\ell), \mathcal{P}(\ell))$*;*

(ii) each edge of $\mathcal{G}_{\text{disk}}(r)$ *at* $\mathcal{P}(\ell)$ *is maintained after the motion step, that is, if* $\|p^{[i]}(\ell)-p^{[j]}(\ell)\|_2 \leq r$*, then also* $\|p^{[i]}(\ell+1)-p^{[j]}(\ell+1)\|_2 \leq r$*;*

(iii) if $\mathcal{G}_{\text{disk}}(r)$ *at time* ℓ *is connected, then* $\mathcal{G}_{\text{disk}}(r)$ *at time* $\ell+1$ *is connected; and*

(iv) the number of connected components of the graph $\mathcal{G}_{\text{disk}}(r)$ *at time* $\ell+1$ *is equal to or smaller than the number of connected components of the graph* $\mathcal{G}_{\text{disk}}(r)$ *at time* ℓ*.*

Remark 4.6 (Constraints for relative-sensing networks: cont'd). Following up on Remark 4.3, the connectivity constraint set in Definition 4.4, written in the frame $\Sigma^{[i]}$, is

$$\mathcal{X}_{\text{disk}}(\mathbf{0}_d, \{p_i^{[1]}, \dots, p_i^{[n]}\}) = \Big\{x \in \overline{B}\Big(\frac{p_i^{[j]}}{2}, \frac{r}{2}\Big) \mid j \neq i \text{ such that } \|p^{[j]} - p^{[i]}\|_2 \leq r\Big\}. \quad \bullet$$

Next, we relax the constraints in Definition 4.4 to provide the network nodes with larger, and therefore less conservative, motion-constraint sets. Recall from Section 2.2 the relative neighborhood graph $\mathcal{G}_{\text{RN}}$, the Gabriel graph $\mathcal{G}_{\text{G}}$, and the r-limited Delaunay graph $\mathcal{G}_{\text{LD}}(r)$. These proximity graphs are illustrated in Figure 2.8. From Theorem 2.8 and Proposition 2.9, respectively, recall that the proximity graphs $\mathcal{G}_{\text{RN}} \cap \mathcal{G}_{\text{disk}}(r)$, $\mathcal{G}_{\text{G}} \cap \mathcal{G}_{\text{disk}}(r)$, and $\mathcal{G}_{\text{LD}}(r)$ have the following properties:

(i) they have the same connected components as $\mathcal{G}_{\text{disk}}(r)$, that is, for all point sets $\mathcal{P} \subset \mathbb{R}^d$, all graphs have the same number of connected components consisting of the same vertices; and

(ii) they are spatially distributed over $\mathcal{G}_{\text{disk}}(r)$.

These mathematical facts have two implications. First, to maintain or decrease the number of connected components of a disk graph, it is sufficient to maintain or decrease the number of connected components of any of the three proximity graphs $\mathcal{G}_{\text{RN}} \cap \mathcal{G}_{\text{disk}}(r)$, $\mathcal{G}_{\text{G}} \cap \mathcal{G}_{\text{disk}}(r)$, and $\mathcal{G}_{\text{LD}}(r)$. Because each of these graphs is more sparse than the disk graph, that is, they are subgraphs of $\mathcal{G}_{\text{disk}}(r)$, fewer connectivity constraints need to be imposed.

Second, because these proximity graphs are spatially distributed over the disk graph, it is possible for each agent to determine which of its neighbors in $\mathcal{G}_{\text{disk}}(r)$ are also its neighbors in these subgraphs. We formalize this discussion as follows.

Definition 4.7 ($\mathcal{G}$-connectivity constraint set). Let $\mathcal{G}$ be a proximity graph that is spatially distributed over $\mathcal{G}_{\text{disk}}(r)$ and that has the same connected components as $\mathcal{G}_{\text{disk}}(r)$. Consider a group of agents at positions $\mathcal{P} = \{p^{[1]}, \dots, p^{[n]}\} \subset \mathbb{R}^d$. The *$\mathcal{G}$-connectivity constraint set* of agent i with respect to $\mathcal{P}$ is

$$\mathcal{X}_{\text{disk},\mathcal{G}}(p^{[i]}, \mathcal{P}) = \big\{x \in \mathcal{X}_{\text{disk}}(p^{[i]}, q) \mid q \in \mathcal{P} \text{ s.t. } (q, p^{[i]}) \text{ is an edge of } \mathcal{G}(\mathcal{P})\big\}. \quad \bullet$$

Lemma 4.8 (Maintaining connectivity of sparser networks). *Let $\mathcal{G}$ be a proximity graph that is spatially distributed over $\mathcal{G}_{\text{disk}}(r)$ and that has the same connected components as $\mathcal{G}_{\text{disk}}(r)$. Consider a group of agents at positions $\mathcal{P}(\ell) = \{p^{[1]}(\ell), \dots, p^{[n]}(\ell)\} \subset \mathbb{R}^d$ at time ℓ. If each agent's control $u^{[i]}(\ell)$ takes value in*

$$u^{[i]}(\ell) \in \mathcal{X}_{\text{disk},\mathcal{G}}\big(p^{[i]}(\ell), \mathcal{P}(\ell)\big) - p^{[i]}(\ell), \quad i \in \{1, \dots, n\},$$

then, according to the discrete-time motion model (4.1.1):

(i) each agent remains in its $\mathcal{G}$-connectivity constraint set;

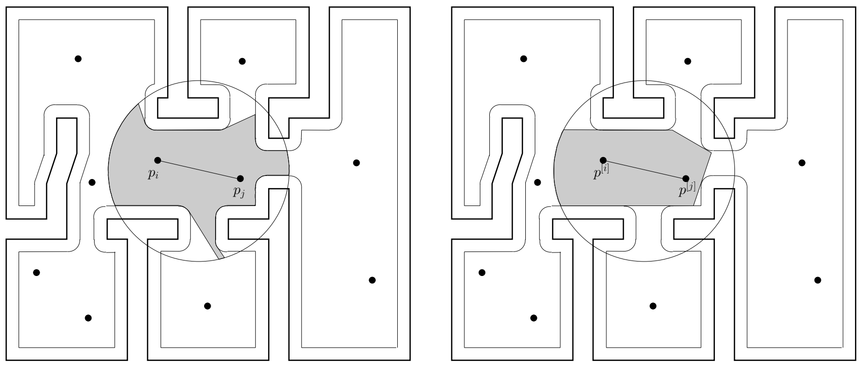
(ii) two agents that are in the same connected component of $\mathcal{G}$ remain at the same connected component after the motion step; and

(iii) the number of connected components of the graph $\mathcal{G}$ at $\mathcal{P}(\ell+1)$ is equal to or smaller than the number of connected components of the graph $\mathcal{G}$ at $\mathcal{P}(\ell)$.

The reader is asked to provide a proof of this result in Exercise E4.1.

4.2.3 Enforcing range-limited line-of-sight links and network connectivity

Here, we consider the connectivity maintenance problem for a group of agents with range-limited line-of-sight communication, as described in Example 3.6. It is convenient to treat directly and only the case of a compact allowable nonconvex environment $Q \subset \mathbb{R}^2$ contracted into $Q_\delta = \{q \in Q \mid \operatorname{dist}(q, \partial Q) \geq \delta\}$ for a small positive δ. We present a solution based on designing constraint sets that guarantee that every edge of the range-limited visibility graph $\mathcal{G}_{\text{vis-disk},Q_\delta}$ is preserved.



(a) (b)

Figure 4.3 Image (a) shows the set $\mathrm{Vi}_{\mathrm{disk}}(p^{[i]}; Q_\delta) \cap \overline{B}(\frac{1}{2}(p^{[i]} + p^{[j]}), \frac{r}{2})$. Image (b) illustrates the execution of the ITERATED TRUNCATION ALGORITHM. Robots i and j are constrained to remain inside the shaded region in (b), which is a convex subset of Q_δ and of the closed ball with center $\frac{1}{2}(p^{[i]} + p^{[j]})$ and radius $\frac{r}{2}$.

We begin with a useful observation and a corresponding geometric algorithm. Assume that, at time ℓ, robot j is inside the range-limited visibility set from $p^{[i]}$ in Q_δ, that is, with the notation of Section 2.1.2,

$$p^{[j]}(\ell) \in \mathrm{Vi}_{\mathrm{disk}}(p^{[i]}(\ell); Q_\delta) = \mathrm{Vi}(p^{[i]}(\ell); Q_\delta) \cap \overline{B}(p^{[i]}(\ell), r).$$

This property holds also at time $\ell + 1$ if $\|p^{[i]}(\ell+1) - p^{[j]}(\ell+1)\|_2 \leq r$ and $[p^{[i]}(\ell+1), p^{[j]}(\ell+1)] \subset Q_\delta$. A sufficient condition is therefore that

$$p^{[i]}(\ell+1),\ p^{[j]}(\ell+1) \in \mathcal{X},$$

for some convex subset $\mathcal{X}$ of $Q_\delta \cap \overline{B}\big(\frac{1}{2}(p^{[i]}(\ell) + p^{[j]}(\ell)), \frac{r}{2}\big)$. Intuitively speaking, $\mathcal{X}$ plays the role of $\mathcal{X}$-constraint set for the proximity graph $\mathcal{G}_{\text{vis-disk},Q_\delta}$. The following geometric algorithm, given the positions $p^{[i]}$ and $p^{[j]}$ in an environment Q_δ, computes precisely one such convex subset:

`function` ITERATED TRUNCATION$(p^{[i]}, p^{[j]}; Q_\delta)$
% Executed by robot i at position $p^{[i]}$ assuming that robot j is at position $p^{[j]}$ within range-limited line of sight of $p^{[i]}$
1: $\mathcal{X}_{\mathrm{temp}} := \mathrm{Vi}_{\mathrm{disk}}(p^{[i]}; Q_\delta) \cap \overline{B}\big(\frac{1}{2}(p^{[i]} + p^{[j]}), \frac{r}{2}\big)$
2: **while** $\partial\mathcal{X}_{\mathrm{temp}}$ contains a concavity **do**
3: $\quad v :=$ a strictly concave point of $\partial\mathcal{X}_{\mathrm{temp}}$ closest to $[p^{[i]}, p^{[j]}]$
4: $\quad \mathcal{X}_{\mathrm{temp}} := \mathcal{X}_{\mathrm{temp}} \cap H_{Q_\delta}(v)$
5: **return** $\mathcal{X}_{\mathrm{temp}}$

Note: in step 3: multiple points belonging to distinct concavities may satisfy the required property. If so, v may be chosen as any of them.

Figure 4.3 illustrates an example convex constraint set computed by the

ITERATED TRUNCATION ALGORITHM. Figure 4.4 illustrates the step-by-step execution required to generate Figure 4.3(b).

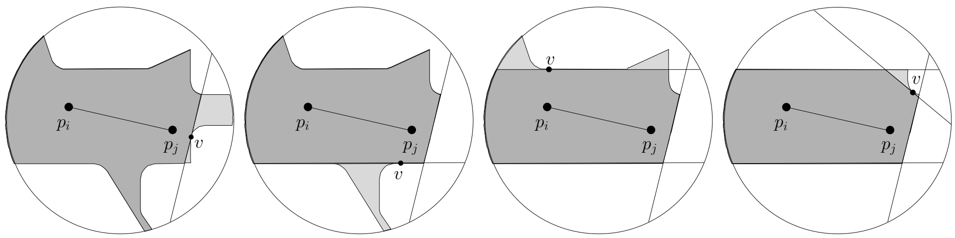


Figure 4.4 From left to right, a sample run of the ITERATED TRUNCATION ALGORITHM. The set $\mathcal{X}_{\text{temp}} := \text{Vi}_{\text{disk}}(p^{[i]}; Q_\delta) \cap \overline{B}(\frac{1}{2}(p^{[i]} + p^{[j]}), \frac{r}{2})$ is shown in Figure 4.3(a). The lightly and darkly shaded regions together represent $\mathcal{X}_{\text{temp}}$ at the current iteration. The darkly shaded region represents $\mathcal{X}_{\text{temp}} \cap H_{Q_\delta}(v)$, where v is as described in step 3:. The outcome of the execution is shown in Figure 4.3(b).

Next, we characterize the main properties of the ITERATED TRUNCATION ALGORITHM. It is convenient to define the set

$$J = \{(p, q) \in Q_\delta \times Q_\delta \mid [p, q] \in Q_\delta \text{ and } \|p - q\|_2 \leq r\}.$$

Proposition 4.9 (Properties of the iterated truncation algorithm). *Consider the δ-contraction of a compact allowable environment Q_δ with κ strict concavities, and let $(p^{[i]}, p^{[j]}) \in J$. The following statements hold:*

(i) The ITERATED TRUNCATION ALGORITHM*, invoked with arguments $(p^{[i]}, p^{[j]}; Q_\delta)$, terminates in at most κ steps; denote its output by $\mathcal{X}_{\text{vis-disk}}(p^{[i]}, p^{[j]}; Q_\delta)$.*

(ii) $\mathcal{X}_{\text{vis-disk}}(p^{[i]}, p^{[j]}; Q_\delta)$ is nonempty, compact and convex.

(iii) $\mathcal{X}_{\text{vis-disk}}(p^{[i]}, p^{[j]}; Q_\delta) = \mathcal{X}_{\text{vis-disk}}(p^{[j]}, p^{[i]}; Q_\delta)$.

(iv) The set-valued map $(p, q) \mapsto \mathcal{X}_{\text{vis-disk}}(p, q; Q_\delta)$ is closed at J.

In the interest of brevity, we do not include the proof here and instead refer the reader to Ganguli et al. (2009). We just mention that fact (iii) is a consequence of the fact that all relevant concavities in the computation of $\mathcal{X}_{\text{vis-disk}}(p^{[i]}, p^{[j]}; Q_\delta)$ are visible from both agents $p^{[i]}$ and $p^{[j]}$. We are finally ready to provide analogs of Definition 4.4 and Lemma 4.5.

Definition 4.10 (Line-of-sight connectivity constraint set). Consider a nonconvex allowable environment Q_δ and two agents i and j within range-limited line of sight. We call $\mathcal{X}_{\text{vis-disk}}(p^{[i]}, p^{[j]}; Q_\delta)$ the *pairwise line-of-sight connectivity constraint set* of agent i with respect to agent j. Furthermore, given agents at positions $\mathcal{P} = \{p^{[1]}, \ldots, p^{[n]}\} \subset Q_\delta$ that are all within range-limited line of sight of agent i, the *line-of-sight connectivity constraint sets* of agent i with respect to $\mathcal{P}$ is

$$\mathcal{X}_{\text{vis-disk}}(p^{[i]}, \mathcal{P}; Q_\delta) = \{x \in \mathcal{X}_{\text{vis-disk}}(p^{[i]}, q; Q_\delta) \mid q \in \mathcal{P} \setminus \{p^{[i]}\}\}. \quad \bullet$$

The following result is a consequence of Proposition 4.9.

Lemma 4.11 (Maintaining network line-of-sight connectivity). *Consider a group of agents at positions $\mathcal{P}(\ell) = \{p^{[1]}(\ell), \dots, p^{[n]}(\ell)\} \subset Q_\delta$ at time ℓ. If each agent's control $u^{[i]}(\ell)$ takes value in*

$$u^{[i]}(\ell) \in \mathcal{X}_{\text{vis-disk}}\big(p^{[i]}(\ell), \mathcal{P}(\ell); Q_\delta\big) - p^{[i]}(\ell), \quad i \in \{1, \dots, n\},$$

then, according to the discrete-time motion model (4.1.1):

(i) each agent remains in its constraint set, that is,

$$p^{[i]}(\ell+1) \in \mathcal{X}_{\text{vis-disk}}(p^{[i]}(\ell), \mathcal{P}(\ell); Q_\delta);$$

(ii) each edge of $\mathcal{G}_{\text{vis-disk},Q_\delta}$ at $\mathcal{P}(\ell)$ is maintained after the motion step, that is, if $p^{[i]}$ and $p^{[j]}$ are within range-limited line of sight at time ℓ, then they are within range-limited line of sight also at time $\ell+1$;

(iii) if $\mathcal{G}_{\text{vis-disk},Q_\delta}$ at $\mathcal{P}(\ell)$ is connected, then $\mathcal{G}_{\text{vis-disk},Q_\delta}$ at $\mathcal{P}(\ell+1)$ is connected; and

(iv) the number of connected components of the graph $\mathcal{G}_{\text{vis-disk},Q_\delta}$ at $\mathcal{P}(\ell+1)$ is equal to or smaller than the number of connected components of the graph $\mathcal{G}_{\text{vis-disk},Q_\delta}$ at $\mathcal{P}(\ell)$.

Remark 4.12 (Constraints for relative-sensing networks: cont'd). Following up on Remarks 4.3 and 4.6, we consider a relative-sensing network with range-limited visibility sensors (see Example 3.16). To compute the connectivity constraint set for this network, it suffices to provide a relative sensing version of the ITERATED TRUNCATION ALGORITHM:

`function` RELATIVE-SENSING ITERATED TRUNCATION$(y; y_{\text{env}})$
% Executed by robot i with range-limited visibility sensor:
robot measurement is $y = p_i^{[j]} \in \text{Vi}_{\text{disk}}(\mathbf{0}_2; (Q_\delta)_i)$ for $j \neq i$
environment measurement is $y_{\text{env}} = \text{Vi}_{\text{disk}}(\mathbf{0}_2; (Q_\delta)_i)$
1: $\mathcal{X}_{\text{temp}} := y_{\text{env}} \cap \overline{B}\big(\frac{p_i^{[j]}}{2}, \frac{r}{2}\big)$
2: **while** $\partial\mathcal{X}_{\text{temp}}$ contains a concavity **do**
3: $v :=$ a strictly concave point of $\partial\mathcal{X}_{\text{temp}}$ closest to $[\mathbf{0}_2, y]$
4: $\mathcal{X}_{\text{temp}} := \mathcal{X}_{\text{temp}} \cap H_{y_{\text{env}}}(v)$
5: **return** $\mathcal{X}_{\text{temp}}$

The algorithm output is $\mathcal{X}_{\text{vis-disk}}(\mathbf{0}_d, y)$, for $y = p_i^{[j]} \in \text{Vi}_{\text{disk}}(\mathbf{0}_2; (Q_\delta)_i)$. •

Next, we relax the constraints in Definition 4.10 to provide the network nodes with larger, and therefore less conservative, motion constraint sets. Similarly to Section 4.2.2, we seek to enforce the preservation of a smaller number of range-limited line-of-sight links, while still making sure that the overall network connectivity is preserved. To do this, we recall from Section 2.2 the notion of locally cliqueless graph $\mathcal{G}_{\text{lc},\mathcal{G}}$ of a proximity graph $\mathcal{G}$. This proximity graph is illustrated in Figure 2.12. Let us use the shorthand notation $\mathcal{G}_{\text{lc-vis-disk},Q_\delta} \equiv \mathcal{G}_{\text{lc},\mathcal{G}_{\text{vis-disk},Q_\delta}}$. From Theorems 2.11(ii) and (iii), respectively, recall that $\mathcal{G}_{\text{lc-vis-disk},Q_\delta}$ has the following properties:

(i) it has the same connected components as $\mathcal{G}_{\text{vis-disk},Q_\delta}$, that is, for all point sets $\mathcal{P} \subset \mathbb{R}^d$, the graph has the same number of connected components consisting of the same vertices; and

(ii) it is spatially distributed over $\mathcal{G}_{\text{vis-disk},Q_\delta}$.

Because of (i), to maintain or decrease the number of connected components of a range-limited visibility graph, it is sufficient to maintain or decrease the number of connected components of the sparser graph $\mathcal{G}_{\text{lc-vis-disk},Q_\delta}$. Because of (ii), it is possible for each agent to determine which of its neighbors in $\mathcal{G}_{\text{vis-disk},Q_\delta}$ are its neighbors also in $\mathcal{G}_{\text{lc-vis-disk},Q_\delta}$. We formalize this discussion as follows.

Definition 4.13 (Locally cliqueless line-of-sight connectivity constraint set). Consider a nonconvex allowable environment $Q_\delta \subset \mathbb{R}^2$ and a group of agents at positions $\mathcal{P} = \{p^{[1]}, \dots, p^{[n]}\} \subset Q$. The *locally cliqueless line-of-sight connectivity constraint set* of agent i with respect to $\mathcal{P}$ is

$$\mathcal{X}_{\text{lc-vis-disk}}(p^{[i]}, \mathcal{P}; Q_\delta) = \big\{x \in \mathcal{X}_{\text{vis-disk}}(p^{[i]}, q; Q_\delta) \mid \\ q \in \mathcal{P} \text{ s.t. } (q, p^{[i]}) \text{ is an edge of } \mathcal{G}_{\text{lc-vis-disk},Q_\delta}(\mathcal{P})\big\}. \quad \bullet$$

The following result is a direct consequence of the previous arguments.

Lemma 4.14 (Maintaining connectivity of sparser networks). *Consider a group of agents at positions $\mathcal{P}(\ell) = \{p^{[1]}(\ell), \dots, p^{[n]}(\ell)\} \subset Q_\delta$ at time ℓ. If each agent's control $u^{[i]}(\ell)$ takes value in*

$$u^{[i]}(\ell) \in \mathcal{X}_{\text{lc-vis-disk}}\big(p^{[i]}(\ell), \mathcal{P}(\ell); Q_\delta\big) - p^{[i]}(\ell), \quad i \in \{1, \dots, n\},$$

then, according to the discrete-time motion model (4.1.1):

(i) each agent remains in its locally cliqueless line-of-sight connectivity constraint set;

(ii) two agents that are in the same connected component of $\mathcal{G}_{\text{lc-vis-disk},Q_\delta}$ remain at the same connected component after the motion step; and

(iii) the number of connected components of the graph $\mathcal{G}_{\text{lc-vis-disk},Q_\delta}$ at

$\mathcal{P}(\ell+1)$ is equal to or smaller than the number of connected components of the graph $\mathcal{G}_{\text{lc-vis-disk},Q_\delta}$ at $\mathcal{P}(\ell)$.

4.3 RENDEZVOUS ALGORITHMS

In this section, we present some algorithms that might be used by a robotic network to achieve rendezvous. Throughout the section, we mainly focus on the uniform network $\mathcal{S}_{\text{disk}}$ of locally connected first-order agents in $\mathbb{R}^d$; this robotic network was introduced in Example 3.4.

4.3.1 Averaging control and communication law

We first study a behavior in which agents move toward a position computed as the average of the received messages. This law is related to the distributed linear algorithms discussed in Section 1.6 and, in particular, to adjacency-based agreement algorithms and Vicsek's model. This algorithm has also been studied in the context of "opinion dynamics under bounded confidence" and is known in the literature as the Krause model.

We loosely describe the AVERAGING law, which we denote by $\mathcal{CC}_{\text{AVERAGING}}$, as follows:

> *[Informal description]* In each communication round each agent performs the following tasks: (i) it transmits its position and receives its neighbors' positions; (ii) it computes the average of the point set comprised of its neighbors and of itself. Between communication rounds, each robot moves toward the average point that it computed.

We next formulate the algorithm, using the description model of Chapter 3. The law is uniform, static, and data-sampled, with standard message-generation function. (Recall from Definition 3.9 and Remark 3.11 that a control and coordination law (1) is uniform if processor state set, message-generation, state-transition and control functions are the same for each agent; (2) is static if the processor state set is a singleton, i.e., the law requires no memory; (3) is data-sampled if if the control functions are independent of the current position of the robot and depend only upon the robots position at the last sample time.)

`Robotic Network:` $\mathcal{S}_{\text{disk}}$ with motion model (4.1.1) in $\mathbb{R}^d$,
with absolute sensing of own position, and
with communication range r

Distributed Algorithm: AVERAGING
Alphabet: $\mathbb{A} = \mathbb{R}^d \cup \{\texttt{null}\}$

function $\text{msg}(p, i)$
1: **return** p

function $\text{ctl}(p, y)$
1: **return** $\text{avrg}(\{p\} \cup \{p_{\text{rcvd}} \mid p_{\text{rcvd}} \text{ is a non-null message in } y\}) - p$

An implementation of this control and communication law is shown in Figure 4.5 for $d = 1$. Note that, along the evolution, (1) several robots *rendezvous*, that is, agree upon a common location, and (2) some robots are connected at the simulation's beginning and not connected at the simulation's end (e.g., robots number 8 and 9, counting from the left). Our analysis

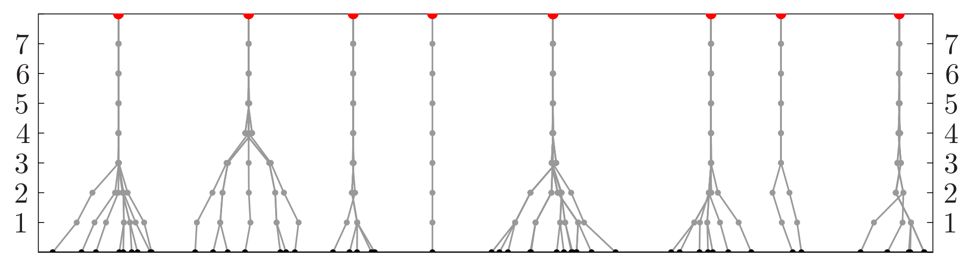


Figure 4.5 The evolution of a robotic network $\mathcal{S}_{\text{disk}}$, with $r = 1.5$, under the AVERAGING control and communication law. The vertical axis corresponds to the elapsed time, and the horizontal axis to the positions of the agents in the real line. The 51 agents are initially randomly deployed over the interval $[-15, 15]$.

of the performance of this law is contained in the following theorem, whose proof is postponed to Section 4.6.1.

Theorem 4.15 (Correctness and time complexity of averaging law). *For $d = 1$, the network $\mathcal{S}_{\text{disk}}$, the law $\mathcal{CC}_{\text{AVERAGING}}$ achieves the task $\mathcal{T}_{\text{rndzvs}}$ with time complexity*

$$\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{AVERAGING}}) \in O(n^5),$$
$$\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{AVERAGING}}) \in \Omega(n).$$

4.3.2 Circumcenter control and communication laws

Here, we define the CRCMCNTR control and communication law for the network $\mathcal{S}_{\text{disk}}$. The law solves the rendezvous problem while keeping the network connected. This law was introduced by Ando et al. (1999) and later studied by Lin et al. (2007a) and Cortés et al. (2006).

We begin by recalling two useful geometric concepts: (i) given a bounded set S, its circumcenter $\mathrm{CC}(S)$ is the center of the closed ball of minimum radius containing S (see Section 2.1.3); (ii) given a point p in a convex set Q and a second point q, the from-to-inside map $\mathrm{fti}(p, q, S)$ is the point in the closed segment $[p, q]$ which is at the same time closest to q and inside S (see Section 2.1.1). Finally, recall also the connectivity constraint set introduced in Definition 4.4.

We loosely describe the CRCMCNTR law, denoted by $\mathcal{CC}_{\text{CRCMCNTR}}$, as follows:

> *[Informal description]* In each communication round each agent performs the following tasks: (i) it transmits its position and receives its neighbors' positions; (ii) it computes the circumcenter of the point set comprised of its neighbors and of itself. Between communication rounds, each robot moves toward this circumcenter point while maintaining connectivity with its neighbors using appropriate connectivity constraint sets.

We next formulate the algorithm, using the description model of Chapter 3. The law is uniform, static, and data-sampled, with standard message-generation function:

`Robotic Network:` $\mathcal{S}_{\text{disk}}$ with discrete-time motion model (4.1.1), with absolute sensing of own position, and with communication range r, in $\mathbb{R}^d$

`Distributed Algorithm:` CRCMCNTR

`Alphabet:` $\mathbb{A} = \mathbb{R}^d \cup \{$`null`$\}$

`function` $\mathrm{msg}(p, i)$

1: **return** p

`function` $\mathrm{ctl}(p, y)$

1: $p_{\text{goal}} := \mathrm{CC}(\{p\} \cup \{p_{\text{rcvd}} \mid \text{for all non-null } p_{\text{rcvd}} \in y\})$
2: $\mathcal{X} := \mathcal{X}_{\text{disk}}(p, \{p_{\text{rcvd}} \mid \text{for all non-null } p_{\text{rcvd}} \in y\})$
3: **return** $\mathrm{fti}(p, p_{\text{goal}}, \mathcal{X}) - p$

This algorithm is illustrated in Figure 4.6.

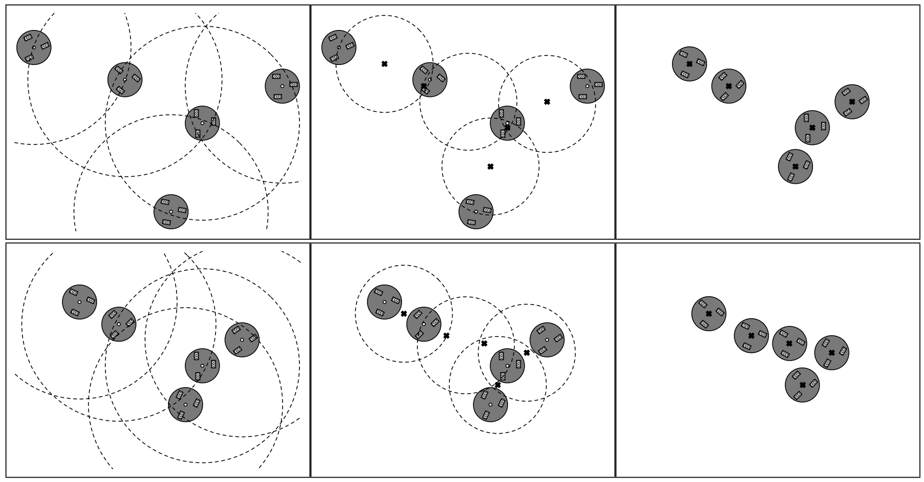


Figure 4.6 An illustration of the execution of the CRCMCNTR ALGORITHM. Each row of plots represents an iteration of the law. In each round, each agent computes its goal point and its constraint set, and then moves toward the goal while remaining in the constraint set.

Next, let us note that it is possible and straightforward to implement the circumcenter law as a static relative-sensing control law on the relative-sensing network with disk sensors $\mathcal{S}^{\mathrm{rs}}_{\mathrm{disk}}$ introduced in Example 3.15:

`Relative Sensing Network:` $\mathcal{S}^{\mathrm{rs}}_{\mathrm{disk}}$ with motion model (4.1.2),
no communication, relative sensing for robot i given by:
robot measurements y contains $p_i^{[j]} \in \overline{B}(\mathbf{0}_2, r)$ for all $j \neq i$

`Distributed Algorithm:` RELATIVE-SENSING CRCMCNTR

`function` $\mathrm{ctl}(y)$

1: $p_{\mathrm{goal}} := \mathrm{CC}(\{\mathbf{0}_d\} \cup \{p_{\mathrm{snsd}} \mid \text{for all non-null } p_{\mathrm{snsd}} \in y\})$
2: $\mathcal{X} := \mathcal{X}_{\mathrm{disk}}(\mathbf{0}_d, \{p_{\mathrm{snsd}} \mid \text{for all non-null } p_{\mathrm{snsd}} \in y\})$
3: **return** $\mathrm{fti}(\mathbf{0}_d, p_{\mathrm{goal}}, \mathcal{X})$

In the remainder of this section, we generalize the circumcenter law in a number of ways: (i) we modify the constraint set by imposing bounds on the control inputs and by relaxing the connectivity constraint as much as possible, while maintaining connectivity guarantees; and (ii) we implement the circumcenter law on two distinct communication graphs. Let us note that many of these generalized circumcenter laws can also be implemented as relative-sensing control laws; in the interest of brevity, we do not present the details.

4.3.2.1 Circumcenter law with control bounds and relaxed connectivity constraints

First, assume that the agents have a compact input space $U = \overline{B}(\mathbf{0}_d, u_{\max})$, with $u_{\max} \in \mathbb{R}_{>0}$. Additionally, we adopt the relaxed $\mathcal{G}$-connectivity constraint sets as follows. Let $\mathcal{G}$ be a proximity graph that is spatially distributed over $\mathcal{G}_{\text{disk}}(r)$ and that has the same connected components as $\mathcal{G}_{\text{disk}}(r)$; examples include $\mathcal{G}_{\text{RN}} \cap \mathcal{G}_{\text{disk}}(r)$, $\mathcal{G}_{\text{G}} \cap \mathcal{G}_{\text{disk}}(r)$, and $\mathcal{G}_{\text{LD}}(r)$. Recall the $\mathcal{G}$-connectivity constraint set from Definition 4.7. Combining the relaxed connectivity constraint and the control magnitude bound, we redefine the control function in the CRCMCNTR law to be:

function $\text{ctl}(p, y)$
% Includes control bound and relaxed $\mathcal{G}$-connectivity constraint
1: $p_{\text{goal}} := \text{CC}(\{p\} \cup \{p_{\text{rcvd}} \mid \text{for all non-null } p_{\text{rcvd}} \in y\})$
2: $\mathcal{X} := \mathcal{X}_{\text{disk},\mathcal{G}}(p, \{p_{\text{rcvd}} \mid \text{for all non-null } p_{\text{rcvd}} \in y\}) \cap \overline{B}(p, u_{\max})$
3: **return** $\text{fti}(p, p_{\text{goal}}, \mathcal{X}) - p$

Second, the circumcenter law can be implemented also on robotic networks with different proximity graphs. For example, we can implement the circumcenter algorithm without any change on the following network.

4.3.2.2 Circumcenter law on the limited Delaunay graph

We consider the same set of physical agents as in $\mathcal{S}_{\text{disk}}$. For $r \in \mathbb{R}_{>0}$, we adopt as communication graph the *r-limited Delaunay graph* $\mathcal{G}_{\text{LD}}(r)$, described in Section 2.2. These data define the uniform robotic network $\mathcal{S}_{\text{LD}} = (\{1, \dots, n\}, \mathcal{R}, E_{\text{LD}})$, as described in Example 3.4. On this network, we implement the CRCMCNTR law without any change, that is, with the same message-generation and control function as we did for the implementation on the network $\mathcal{S}_{\text{disk}}$.

4.3.2.3 Parallel circumcenter law on the ∞-disk graph

We consider the network $\mathcal{S}_{\infty\text{-disk}}$ of first-order robots in $\mathbb{R}^d$, connected according to the $\mathcal{G}_{\infty\text{-disk}}(r)$ graph (see Example 3.4). For this network, we define the PLL-CRCMCNTR law, which we denoted by $\mathcal{CC}_{\text{PLL-CRCMCNTR}}$, by designing d decoupled circumcenter laws running in parallel on each coordinate axis of $\mathbb{R}^d$. As before, this law is uniform and static. What is remarkable, however, is that no constraint is required to maintain connectivity (see Exercise E4.4).

The parallel circumcenter of the set S, denoted by $\mathrm{PCC}(S)$, is the center of the smallest axis-aligned rectangle containing S. In other words, $\mathrm{PCC}(S)$ is the component-wise circumcenter of S (see Figure 4.7). We state the

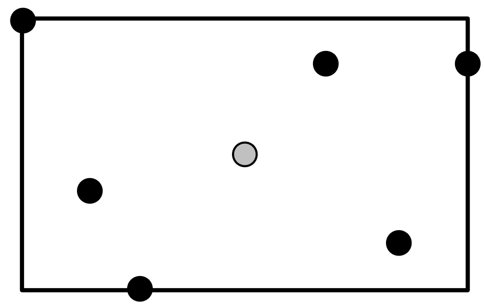


Figure 4.7 The gray point is the parallel circumcenter of the collection of black points.

parallel circumcenter law as follows:

`Robotic Network:` $\mathcal{S}_{\infty\text{-disk}}$ with discrete-time motion model (4.1.1) in $\mathbb{R}^d$, with absolute sensing of own position, and with communication range r in L^∞-metric

`Distributed Algorithm:` PLL-CRCMCNTR

`Alphabet:` $\mathbb{A} = \mathbb{R}^d \cup \{\texttt{null}\}$

`function` $\mathrm{msg}(p, i)$

1: **return** p

`function` $\mathrm{ctl}(p, y)$

1: $p_{\mathrm{goal}} := \mathrm{PCC}(\{p\} \cup \{p_{\mathrm{rcvd}} \mid \text{for all non-null } p_{\mathrm{rcvd}} \in y\})$
2: **return** $p_{\mathrm{goal}} - p$

4.3.3 Correctness and complexity of circumcenter laws

In this section, we characterize the convergence and complexity properties of the circumcenter law and of its variations. The following theorem summarizes the results known in the literature about the asymptotic properties of the circumcenter law.

Theorem 4.16 (Correctness of the circumcenter laws). *For $d \in \mathbb{N}$, $r \in \mathbb{R}_{>0}$, and $\varepsilon \in \mathbb{R}_{>0}$, the following statements hold:*

(i) *on the network* $\mathcal{S}_{\text{disk}}$*, the law* $\mathcal{CC}_{\text{CRCMCNTR}}$ *(with control magnitude bounds and relaxed* $\mathcal{G}$*-connectivity constraints) achieves the exact rendezvous task* $\mathcal{T}_{\text{rndzvs}}$*;*

(ii) *on the network* $\mathcal{S}_{\text{LD}}$*, the law* $\mathcal{CC}_{\text{CRCMCNTR}}$ *achieves the* ε*-rendezvous task* $\mathcal{T}_{\varepsilon\text{-rndzvs}}$*; and*

(iii) *on the network* $\mathcal{S}_{\infty\text{-disk}}$*, the law* $\mathcal{CC}_{\text{PLL-CRCMCNTR}}$ *achieves the exact rendezvous task* $\mathcal{T}_{\text{rndzvs}}$*.*

Furthermore, the evolutions of $(\mathcal{S}_{\text{disk}}, \mathcal{CC}_{\text{CRCMCNTR}})$*,* $(\mathcal{S}_{\text{LD}}, \mathcal{CC}_{\text{CRCMCNTR}})$*, and* $(\mathcal{S}_{\infty\text{-disk}}, \mathcal{CC}_{\text{PLL-CRCMCNTR}})$ *have the following properties:*

(iv) *If any two agents belong to the same connected component of the respective communication graph at* $\ell \in \mathbb{Z}_{\geq 0}$*, then they continue to belong to the same connected component for all subsequent times* $k \geq \ell$*.*

(v) *For each evolution, there exists* $P^* = (p_1^*, \dots, p_n^*) \in (\mathbb{R}^d)^n$ *such that:*

  (a) *the evolution asymptotically approaches* P^**; and*

  (b) *for each* $i, j \in \{1, \dots, n\}$*, either* $p_i^* = p_j^*$*, or* $\|p_i^* - p_j^*\|_2 > r$ *(for the networks* $\mathcal{S}_{\text{disk}}$ *and* $\mathcal{S}_{\text{LD}}$*) or* $\|p_i^* - p_j^*\|_\infty > r$ *(for the network* $\mathcal{S}_{\infty\text{-disk}}$*).*

The proof of this theorem is given in Section 4.6.2. The robustness of the circumcenter control and communication laws can be characterized with respect to link failures (see Cortés et al., 2006).

Next, we analyze the time complexity of $\mathcal{CC}_{\text{CRCMCNTR}}$. As we will see, next, the complexity of $\mathcal{CC}_{\text{CRCMCNTR}}$ differs dramatically when applied to robotic networks with different communication graphs. We provide complete results for the networks $\mathcal{S}_{\text{disk}}$ and $\mathcal{S}_{\text{LD}}$ only for the case $d = 1$.

Theorem 4.17 (Time complexity of circumcenter laws). *For* $r \in \mathbb{R}_{>0}$ *and* $\varepsilon \in \,]0, 1[$*, the following statements hold:*

(i) *on the network* $\mathcal{S}_{\text{disk}}$*, evolving on the real line* $\mathbb{R}$ *(i.e., with* $d = 1$*),* $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{CRCMCNTR}}) \in \Theta(n)$*;*

(ii) *on the network* $\mathcal{S}_{\text{LD}}$*, evolving on the real line* $\mathbb{R}$ *(i.e., with* $d = 1$*),* $\text{TC}(\mathcal{T}_{(r\varepsilon)\text{-rndzvs}}, \mathcal{CC}_{\text{CRCMCNTR}}) \in \Theta(n^2 \log(n\varepsilon^{-1}))$*; and*

(iii) *on the network* $\mathcal{S}_{\infty\text{-disk}}$*, evolving on Euclidean space (i.e., with* $d \in \mathbb{N}$*),* $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{PLL-CRCMCNTR}}) \in \Theta(n)$*.*

The proof of this result is contained in Martínez et al. (2007).

Remark 4.18 (Analysis in higher dimensions). The results in Theorems 4.17(i) and (ii) induce lower bounds on the time complexity of the circumcenter law in higher dimensions. Indeed, for arbitrary $d \geq 1$, we have the following:

(i) on the network $\mathcal{S}_{\text{disk}}$, $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{CRCMCNTR}}) \in \Omega(n)$;

(ii) on the network $\mathcal{S}_{\text{LD}}$, $\text{TC}(\mathcal{T}_{(r\varepsilon)\text{-rndzvs}}, \mathcal{CC}_{\text{CRCMCNTR}}) \in \Omega(n^2 \log(n\varepsilon^{-1}))$.

We have performed extensive numerical simulations for the case $d = 2$ and the network $\mathcal{S}_{\text{disk}}$. We run the algorithm starting from generic initial configurations (where, in particular, the robots' positions are not aligned) contained in a bounded region of $\mathbb{R}^2$. We have consistently obtained that the time complexity to achieve $\mathcal{T}_{\text{rndzvs}}$ with $\mathcal{CC}_{\text{CRCMCNTR}}$ starting from these initial configurations is independent of the number of robots. This leads us to conjecture that initial configurations where all robots are aligned (equivalently, the 1-dimensional case) give rise to the worst possible performance of the algorithm. In other words, we conjecture that, for $d \geq 2$, $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{CRCMCNTR}}) = \Theta(n)$. •

Remark 4.19 (Congestion effects). As discussed in Remark 3.8, one way of incorporating congestion effects into the network operation is to assume that the parameters of the physical components of the network depend upon the number of robots—for instance, by assuming that the communication range decreases with the number of robots. Theorem 4.17 presents an alternative, equivalent, way of looking at congestion: the results hold under the assumption that the communication range is constant, but allow for the diameter of the initial network configuration (the maximum inter-agent distance) to grow unbounded with the number of robots. •

4.3.4 The circumcenter law in nonconvex environments

In this section, we adapt the circumcenter algorithm to work on networks in planar nonconvex allowable environments. Throughout the section, we only consider the case of a compact allowable nonconvex environment Q contracted into Q_δ for a small positive δ. We present the algorithm in two formats: for the communication-based network $\mathcal{S}_{\text{vis-disk}}$ described in Example 3.6, and for the relative-sensing network $\mathcal{S}^{\text{rs}}_{\text{vis-disk}}$ described in Example 3.16.

We modify the circumcenter algorithm in three ways: first, we adopt the connectivity constraints described in the previous section for range-limited line-of-sight links; second, we further restrict the robot motion to remain inside the relative convex hull of the sensed robot positions; and third, we move towards the circumcenter of the constraint set, instead of the circumcenter of the neighbors positions. The details of the algorithm are as follows:

`Robotic Network:` $\mathcal{S}_{\text{vis-disk}}$ with discrete-time motion model (4.1.1), absolute sensing of own position and of Q_δ, and communication range r within line of sight ($\mathcal{G}_{\text{vis-disk},Q_\delta}$)

`Distributed Algorithm:` NONCONVEX CRCMCNTR

`Alphabet:` $\mathbb{A} = \mathbb{R}^2 \cup \{\texttt{null}\}$

`function` $\text{msg}(p, i)$
1: **return** p

`function` $\text{ctl}(p, y)$
1: $\mathcal{X}_1 := \mathcal{X}_{\text{vis-disk}}(p, \{p_{\text{rcvd}} \mid \text{for all non-null } p_{\text{rcvd}} \in y\}; Q_\delta)$
2: $\mathcal{X}_2 := \text{rco}(\{p\} \cup \{p_{\text{rcvd}} \mid \text{for all non-null } p_{\text{rcvd}} \in y\}; \text{Vi}(p; Q_\delta))$
3: $p_{\text{goal}} := \text{CC}(\mathcal{X}_1 \cap \mathcal{X}_2)$
4: **return** $\text{fti}(p, p_{\text{goal}}, \overline{B}(p, u_{\max})) - p$

Next, we present the relative sensing version; recall that $p_i^{[i]} = \mathbf{0}_2$ and that, as discussed in Section 3.2.3 in the context of the evolution of a relative sensing network with environment sensors, y_{env} denotes the environment measurement provided by the range-limited visibility sensor:

`Relative Sensing Network:` $\mathcal{S}^{\text{rs}}_{\text{vis-disk}}$ with motion model (4.1.2) in Q_δ,
no communication, relative sensing for robot i given by:
robot measurements y contains $p_i^{[j]} \in \text{Vi}_{\text{disk}}(\mathbf{0}_2; (Q_\delta)_i)$ for $j \neq i$
environment sensing is $y_{\text{env}} = \text{Vi}_{\text{disk}}(\mathbf{0}_2; (Q_\delta)_i)$

`Distributed Algorithm:` NONCONVEX RELATIVE-SENSING CRCMCNTR

`function` $\text{ctl}(y, y_{\text{env}})$
1: $\mathcal{X}_1 := \mathcal{X}_{\text{vis-disk}}(\mathbf{0}_2, \{p_{\text{snsd}} \mid \text{for all non-null } p_{\text{snsd}} \in y\}; y_{\text{env}})$
2: $\mathcal{X}_2 := \text{rco}(\{\mathbf{0}_2\} \cup \{p_{\text{snsd}} \mid \text{for all non-null } p_{\text{snsd}} \in y\}; y_{\text{env}})$
3: $p_{\text{goal}} := \text{CC}(\mathcal{X}_1 \cap \mathcal{X}_2)$
4: **return** $\text{fti}(\mathbf{0}_2, p_{\text{goal}}, \overline{B}(\mathbf{0}_2, u_{\max}))$

Theorem 4.20 (Correctness of the circumcenter law in nonconvex environments). *For $\delta > 0$, let Q_δ be a contraction of a compact allowable nonconvex environment Q. For $r \in \mathbb{R}_{>0}$ and $\varepsilon \in \mathbb{R}_{>0}$, on the network $\mathcal{S}_{\text{vis-disk}}$, the law $\mathcal{CC}_{\text{NONCONVEX CRCMCNTR}}$ (with control magnitude bounds) achieves the ε-rendezvous task $\mathcal{T}_{\varepsilon\text{-rndzvs}}$. Furthermore, the evolution has the following properties:*

(i) If any two agents belong to the same connected component of the graph $\mathcal{G}_{\text{vis-disk},Q_\delta}$ at $\ell \in \mathbb{Z}_{\geq 0}$, then they continue to belong to the same connected component for all subsequent times $k \geq \ell$.

(ii) There exists $P^* = (p_1^*, \dots, p_n^*) \in Q_\delta^n$ such that:

(a) the evolution asymptotically approaches P^*; and

(b) for each $i, j \in \{1, \dots, n\}$, either $p_i^* = p_j^*$, or p_i^* and p_j^* are not within range-limited line of sight.

The proof of this result can be found in Ganguli et al. (2009). A brief sketch of the proof steps is presented in Section 4.6.4. The complexity of the NONCONVEX CRCMCNTR law has not been characterized. However, note that the evolution from any initial configuration such that $\mathcal{G}_{\text{vis},Q_\delta}$ is complete is also an evolution of the CRCMCNTR law discussed in Section 4.3.2, and hence Theorem 4.17(i) induces a lower bound on the time complexity.

4.4 SIMULATION RESULTS

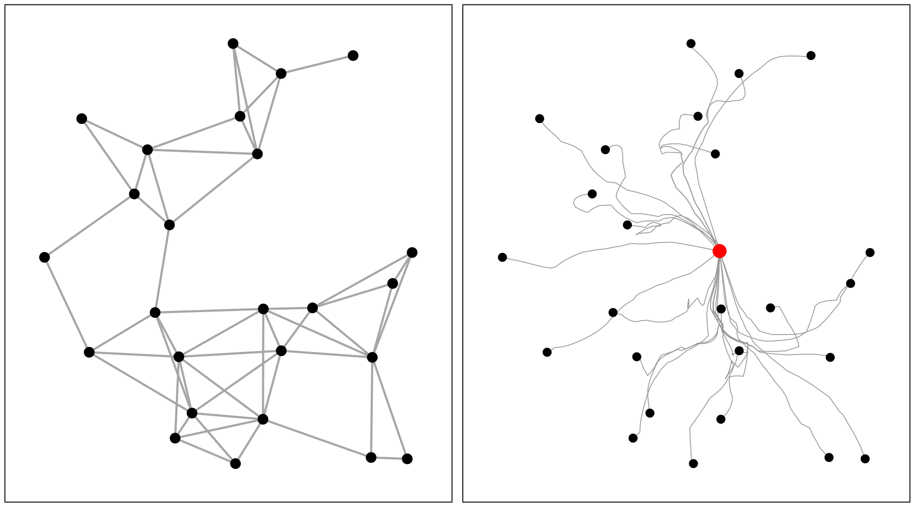
In this section, we illustrate the execution of some circumcenter control and communication laws introduced in this chapter. The CRCMCNTR law is implemented on the networks $\mathcal{S}_{\text{disk}}$, $\mathcal{S}_{\text{LD}}$, and $\mathcal{S}_{\infty\text{-disk}}$ in Mathematica® as a library of routines and a main program running the simulation. The packages `PlanGeom.m` and `SpatialGeom.m` contain routines for the computation of geometric objects in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. These routines are freely available at the book webpage http://coordinationbook.info

First, we show evolutions of $(\mathcal{S}_{\text{disk}}, \text{CRCMCNTR})$ in two and three dimensions in Figures 4.8 and 4.9, respectively. Measuring displacements in meters, we consider random initial positions over the square $[-7, 7] \times [-7, 7]$ and the cube $[-7, 7] \times [-7, 7] \times [-7, 7]$. The 25 robotic agents have a communication radius $r = 4$ and a compact input space $U = \overline{B}(\mathbf{0}_d, u_{\max})$, with $u_{\max} = 0.15$. As the simulations show, the task $\mathcal{T}_{\text{rndzvs}}$ is achieved, as guaranteed by Theorem 4.16(i).

Second, within the same setup, we show an evolution of $(\mathcal{S}_{\text{LD}}, \text{CRCMCNTR})$ in two dimensions in Figure 4.10. As the simulation shows, the task $\mathcal{T}_{\varepsilon\text{-rndzvs}}$ is achieved, as guaranteed by Theorem 4.16(ii).

Third, we show an evolution of $(\mathcal{S}_{\infty\text{-disk}}, \text{PLL-CRCMCNTR})$ in two dimensions in Figure 4.11. As the simulations show, the task $\mathcal{T}_{\text{rndzvs}}$ is achieved, as guaranteed by Theorem 4.16(iii).

Finally, we refer the interested reader to Ganguli et al. (2009) for simulation results for the NONCONVEX CRCMCNTR algorithm.



(a) (b)

Figure 4.8 The evolution of ($\mathcal{S}_{\text{disk}}$, CRCMCNTR) with $n = 25$ robots in 2 dimensions: (a) shows the initial connected network configuration; (b) shows the evolution of the individual agents until rendezvous is achieved.

4.5 NOTES

The rendezvous problem and the circumcenter algorithm were originally introduced by Ando et al. (1999). The circumcenter algorithm has been extended to other control policies, including asynchronous implementations, in Lin et al. (2007a,b). The circumcenter algorithm has been extended beyond planar problems to arbitrary dimensions in Cortés et al. (2006), where its robustness properties are also characterized. Regarding Theorem 4.16, the results on $\mathcal{S}_{\text{disk}}$ appeared originally in Ando et al. (1999); the results on $\mathcal{S}_{\text{LD}}$ and on $\mathcal{S}_{\infty\text{-disk}}$ appeared originally in Cortés et al. (2006) and in Martínez et al. (2007), respectively. Variations of the circumcenter law in the presence of noise and sensor errors are studied in Martínez (2009). The continuous-time version of the circumcenter law, with no connectivity constraints, is analyzed in Lin et al. (2007c). Continuous-time control laws for groups of robots with simple first-order dynamics and unicycle dynamics are proposed in Lin et al. (2004, 2005) and Dimarogonas and Kyriakopoulos (2007). In these works, the inter-robot topology is time dependent and assumed *a priori* to be connected at all times. Rendezvous under communication quantization is studied in Fagnani et al. (2004) and Carli and Bullo (2009). Rendezvous for unicycle robots with minimal sensing capabilities is studied by Yu et al. (2008). Relationships with classic curve-shortening flows are studied by Smith et al. (2007).

Rendezvous has also been studied within the computer science literature,

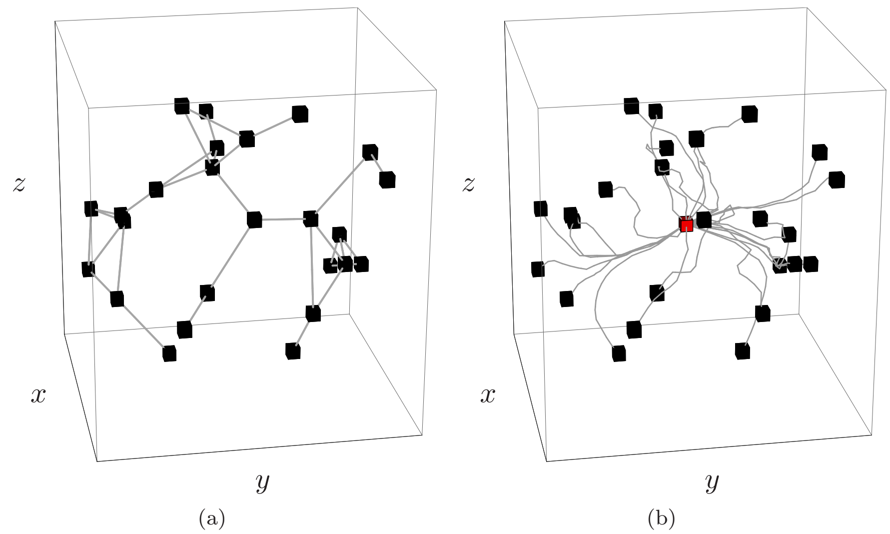
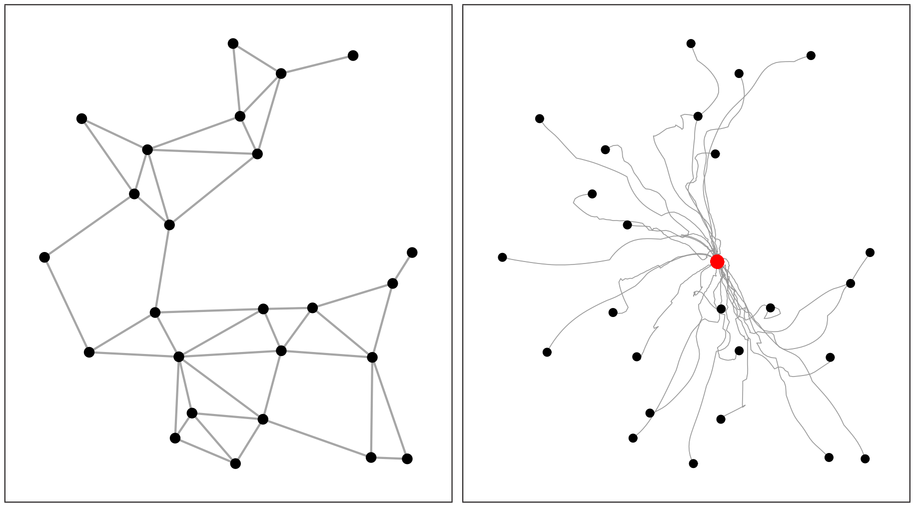


Figure 4.9 The evolution of ($\mathcal{S}_{\text{disk}}$, CRCMCNTR) with $n = 25$ robots in 3 dimensions: (a) shows the initial connected network configuration; (b) shows the evolution of the individual agents until rendezvous is achieved.

where the problem is referred to as the "gathering," or point formation, problem. Flocchini et al. (1999) and Suzuki and Yamashita (1999) study the point formation problem under the assumption that each robot is capable of sensing all other robots. Flocchini et al. (2005) propose asynchronous algorithms to solve the gathering problem, and Agmon and Peleg (2006) study the solvability of the problem in the presence of faulty robots.

Multi-robot rendezvous with line-of-sight sensors is considered in Roy and Dudek (2001), where solutions are proposed based on the exploration of the unknown environment and the selection of appropriate rendezvous points at pre-specified times. Hayes et al. (2003) also consider rendezvous at a specified location for visually guided agents, but the proposed solution requires each agent to have knowledge of the location of all other agents. The problem of computing a multi-robot rendezvous point in polyhedral surfaces made of triangular faces is considered in Lanthier et al. (2005). The perimeter-minimizing algorithm presented by Ganguli et al. (2009) solves the rendezvous problem for sensor-based networks with line-of-sight range-limited sensors in nonconvex environments.

Regarding the connectivity maintenance problem, a number of works have addressed the problem of designing a coordination algorithm that achieves a general, non-specified task while preserving connectivity. The centralized solution proposed in Zavlanos and Pappas (2005) allows for a general range of agent motions. The distributed solution presented by Savla et al. (2009)

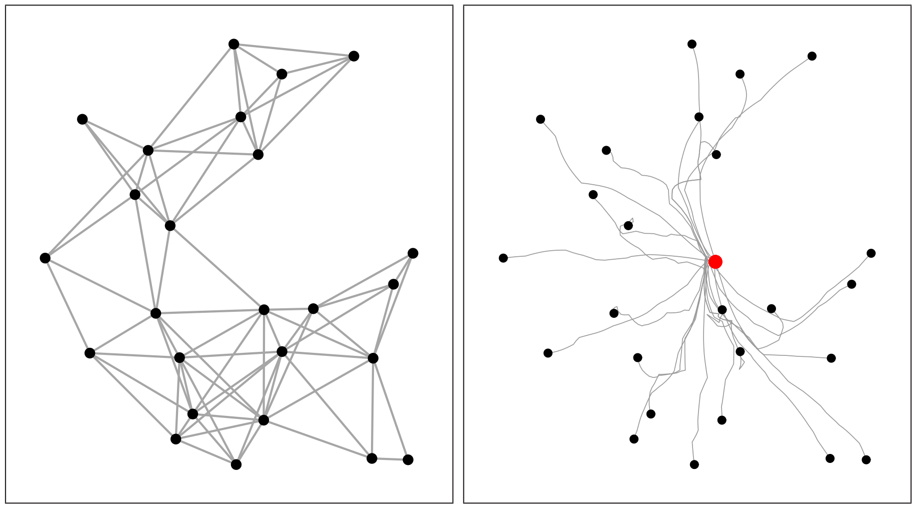


(a) (b)

Figure 4.10 The evolution of $(\mathcal{S}_{\mathrm{LD}}, \mathrm{CRCMCNTR})$ with $n = 25$ robots in 2 dimensions: (a) shows the initial connected network configuration; (b) shows the evolution of the individual agents until rendezvous is achieved.

gives connectivity maintaining constraints for second-order control systems with input magnitude bounds. A distributed algorithm to perform graph rearrangements that preserve the connectivity is presented in Schuresko and Cortés (2007). Connectivity problems have been studied also in other contexts. Langbort and Gupta (2009) study the impact of the connectivity of the interconnection topology in a class of network optimization problems. Spanos and Murray (2005) generate connectivity-preserving motions between pairs of formations. Ji and Egerstedt (2007) design Laplacian-based control laws to solve formation control problems while preserving connectivity. Various works have focused on designing the network motion so that some desired measure of connectivity (e.g., algebraic connectivity) is maximized under position constraints. Boyd (2006) and de Gennaro and Jadbabaie (2006) consider convex constraints, while Kim and Mesbahi (2006) deal with a class of nonconvex constraints. Zavlanos and Pappas (2007) use potential fields to maximize algebraic connectivity.

A continuous-time version of the averaging control and communication law is also known as the Hegselmann-Krause model for "opinion dynamics under bounded confidence" (see Hegselmann and Krause, 2002; Lorenz, 2007). In this model, each agent may change its opinion by averaging it with that of neighbors who are in an ε-confidence area. In other words, the difference between the agent's opinion and those of its neighbors' should be bounded by ε. A similar model where the communication between agents is random is the Deffuant-Weisbuch model, inspired by a model of dissemination of



(a) (b)

Figure 4.11 The evolution of $(\mathcal{S}_{\infty\text{-disk}}, \text{PLL-CRCMCNTR})$ with $n = 25$ robots in 2 dimensions: (a) shows the initial connected network configuration; (b) shows the evolution of the individual agents until rendezvous is achieved.

culture (see Deffuant et al., 2000; Axelrod, 1997).

4.6 PROOFS

This section gathers the proofs of the main results presented in the chapter.

4.6.1 Proof of Theorem 4.15

Proof. One can easily prove that, along the evolution of the network, the ordering of the agents is preserved, that is, the inequality $p^{[i]} \leq p^{[j]}$ is preserved at the next time step. However, links between agents are not necessarily preserved (see, e.g., Figure 4.8). Indeed, connected components may split along the evolution. However, merging events do not occur. Consider two contiguous connected components C_1 and C_2 of $\mathcal{G}_{\text{disk}}(r)$, with C_1 to the left of C_2. By definition, the rightmost agent in the component C_1 and the leftmost agent in the component C_2 are at a distance strictly larger than r. Now, by executing the algorithm, they can only but increase that distance, since the rightmost agent in C_1 will move to the left, and the leftmost agent in C_2 will move to the right. Therefore, connected components do not merge.

Consider first the case of an initial network configuration for which the communication graph remains connected throughout the evolution. Without loss of generality, assume that the agents are ordered from left to right according to their identifier, that is, $p^{[1]}(0) \leq \cdots \leq p^{[n]}(0)$. Let $\alpha \in \{3, \dots, n\}$ have the property that agents $\{2, \dots, \alpha - 1\}$ are neighbors of agent 1, and agent α is not. (If, instead, all agents are within an interval of length r, then rendezvous is achieved in 1 time instant, and the statement in theorem is easily seen to be true.) Note that we can assume that agents $\{2, \dots, \alpha - 1\}$ are also neighbors of agent α. If this is not the case, then those agents that are neighbors of agent 1 and not of agent α rendezvous with agent 1 at the next time instant. At the time instant $\ell = 1$, the new updated positions satisfy

$$
\begin{aligned}
p^{[1]}(1) &= \frac{1}{\alpha - 1} \sum_{k=1}^{\alpha - 1} p^{[k]}(0), \\
p^{[\gamma]}(1) &\in \Big[\frac{1}{\alpha} \sum_{k=1}^{\alpha} p^{[k]}(0), *\Big], \ \gamma \in \{2, \dots, \alpha - 1\},
\end{aligned}
$$

where $*$ denotes a certain unimportant point.

Now, we show that

$$p^{[1]}(\alpha - 1) - p^{[1]}(0) \geq \frac{r}{\alpha(\alpha - 1)}. \tag{4.6.1}$$

Let us first show the inequality for $\alpha = 3$. Because of the assumption that the communication graph remains connected, agent 2 is still a neighbor of agent 1 at the time instant $\ell = 1$. Therefore, $p^{[1]}(2) \geq \frac{1}{2}(p^{[1]}(1) + p^{[2]}(1))$, and from here we deduce

$$
\begin{aligned}
&p^{[1]}(2) - p^{[1]}(0) \geq \frac{1}{2}\big(p^{[2]}(1) - p^{[1]}(0)\big) \\
&\quad \geq \frac{1}{2}\big(\frac{1}{3}\big(p^{[1]}(0) + p^{[2]}(0) + p^{[3]}(0)\big) - p^{[1]}(0)\big) \geq \frac{1}{6}\big(p^{[3]}(0) - p^{[1]}(0)\big) \geq \frac{r}{6}.
\end{aligned}
$$

Let us now proceed by induction. Assume that inequality (4.6.1) is valid for $\alpha - 1$, and let us prove it for α. Consider first the possibility, when at the time instant $\ell = 1$, that the agent $\alpha - 1$ is still a neighbor of agent 1. In this case, $p^{[1]}(2) \geq \frac{1}{\alpha - 1} \sum_{k=1}^{\alpha - 1} p^{[k]}(1)$, and from here we deduce

$$
\begin{aligned}
p^{[1]}(2) - p^{[1]}(0) &\geq \frac{1}{\alpha - 1}\Big(p^{[\alpha - 1]}(1) - p^{[1]}(0)\Big) \\
&\geq \frac{1}{\alpha - 1}\Big(\frac{1}{\alpha} \sum_{k=1}^{\alpha} p^{[k]}(0) - p^{[1]}(0)\Big) \\
&\geq \frac{1}{\alpha(\alpha - 1)}\Big(p^{[\alpha]}(0) - p^{[1]}(0)\Big) \geq \frac{r}{\alpha(\alpha - 1)},
\end{aligned}
$$

which, in particular, implies (4.6.1). Consider then the case when agent $\alpha - 1$ is not a neighbor of agent 1 at the time instant $\ell = 1$. Let $\beta < \alpha$

such that agent $\beta - 1$ is a neighbor of agent 1 at $\ell = 1$, but agent β is not. Since $\beta < \alpha$, we have by induction $p^{[1]}(\beta) - p^{[1]}(1) \geq \frac{r}{\beta(\beta-1)}$. From here, we deduce that $p^{[1]}(\alpha - 1) - p^{[1]}(0) \geq \frac{r}{\alpha(\alpha-1)}$.

It is clear that after $\ell_1 = \alpha - 1$, we could again consider two complementary cases (either agent 1 has all others as neighbors or not) and repeat the same argument once again. In that way, we would find ℓ_2 such that the distance traveled by agent 1 after ℓ_2 rounds would be lower bounded by $\frac{2r}{n(n-1)}$. Repeating this argument iteratively, the worst possible case is one in which agent 1 keeps moving to the right and, at each time step, there is always another agent which is not a neighbor. Since the diameter of the initial condition P_0 is upper bounded by $(n-1)r$, in the worst possible situation, there exists some time ℓ_k such that $\frac{kr}{(n-1)n} = O(r(n-1))$. This implies that $k = O((n-1)^2 n)$. Now we can upper bound the total convergence time ℓ_k by $\ell_k = \sum_{i=1}^{k} \alpha_i - k \leq k(n-1)$, where we have used that $\alpha_i \leq n$ for all $i \in \{1, \dots, n\}$. From here, we see that $\ell_k = O((n-1)^3 n)$, and hence we deduce that in $O(n(n-1)^3)$ time instants there cannot be any agent which is not a neighbor of the agent 1. Hence, all agents rendezvous at the next time instant. Consequently,

$$\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{AVERAGING}}, P_0) = O(n(n-1)^3).$$

Finally, for a general initial configuration P_0, because there are a finite number of agents, only a finite number of splittings (at most $n-1$) of the connected components of the communication graph can take place along the evolution. Therefore, we conclude that $\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{AVERAGING}}) = O(n^5)$.

Let us now prove the lower bound. Consider an initial configuration $P_0 \in \mathbb{R}^n$ where all agents are positioned in increasing order according to their identity, and exactly at a distance r apart—say, $p^{[i+1]}(0) - p^{[i]}(0) = r$, $i \in \{1, \dots, n-1\}$. Assume for simplicity that n is odd—when n is even, one can reason in an analogous way. Because of the symmetry of the initial condition, in the first time step, only agents 1 and n move. All the remaining agents remain in their position, because it coincides with the average of its neighbors' position and its own. At the second time step, only agents 1, 2, $n-1$, and n move, and the others remain static because of the symmetry. Applying this idea iteratively, one deduces that the time step when agents $\frac{n-1}{2}$ and $\frac{n+3}{2}$ move for the first time is lower bounded by $\frac{n-1}{2}$. Since both agents have still at least a neighbor (agent $\frac{n+1}{2}$), the task $\mathcal{T}_{\mathrm{rndzvs}}$ has not been achieved yet at this time step. Therefore, $\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{AVERAGING}}, P_0) \geq \frac{n-1}{2}$, and the result follows. ■

4.6.2 Proof of Theorem 4.16

Proof. We divide the proof of the theorem into three groups, one per network.

STEP 1: Facts on $(\mathcal{S}_{\text{disk}}, \mathcal{CC}_{\text{CRCMCNTR}})$. Fact (iv) for $(\mathcal{S}_{\text{disk}}, \mathcal{CC}_{\text{CRCMCNTR}})$ is a direct consequence of the control function definition of the CRCMCNTR law and Lemma 4.8.

Let us show fact (i). Because $\mathcal{G}$ has the same connected components as $\mathcal{G}_{\text{disk}}(r)$, fact (iv) implies that the number of connected components of $\mathcal{G}_{\text{disk}}(r)$ can only but decrease. In other words, the number of agents in each of the connected components of $\mathcal{G}_{\text{disk}}(r)$ is non-decreasing. Since there is a finite number of agents, there must exist ℓ_0 such that the identity of the agents in each connected component of $\mathcal{G}_{\text{disk}}(r)$ is fixed for all $\ell \geq \ell_0$ (that is, no more agents are added to the connected component afterwards). In what follows, without loss of generality, we assume that there is only one connected component after ℓ_0, i.e., the graph is connected (if this is not the case, then the same argument follows through for each connected component).

We prove that the law $\mathcal{CC}_{\text{CRCMCNTR}}$ (with control magnitude bounds and relaxed $\mathcal{G}$-connectivity constraints) achieves the exact rendezvous task $\mathcal{T}_{\text{rndzvs}}$ in the following two steps:

(a) We first define a set-valued dynamical system $((\mathbb{R}^d)^n, (\mathbb{R}^d)^n, T)$ such that the evolutions of $(\mathcal{S}_{\text{disk}}, \mathcal{CC}_{\text{CRCMCNTR}})$, starting from an initial configuration where $\mathcal{G}_{\text{disk}}(r)$ is connected, are contained in the set of evolutions of the set-valued dynamical system.

(b) We then establish that any evolution of $((\mathbb{R}^d)^n, (\mathbb{R}^d)^n, T)$ converges to a point in $\operatorname{diag}((\mathbb{R}^d)^n)$ (the point might be different for different evolutions).

This strategy is analogous to the discussion regarding the Overapproximation Lemma for time-dependent systems in Section 1.3.5.

Let as perform (a). Given a connected graph G with vertices $\{1, \dots, n\}$, let us consider the constraint sets and goal points defined with respect to G. In other words, given $P = (p_1, \dots, p_n) \in (\mathbb{R}^d)^n$, define for each $i \in \{1, \dots, n\}$,

$$
\begin{aligned}
(p_{\text{goal}})_i &:= \operatorname{CC}(\{p_i\} \cup \{p_j \mid j \in \mathcal{N}_G(i)\}), \\
\mathcal{X}_i &:= \bigcap \left\{ \overline{B}\left(\tfrac{p_i+p_j}{2}, \tfrac{r_i(P)}{2}\right) \mid j \in \mathcal{N}_G(i) \right\} \cap \overline{B}(p_i, u_{\max}),
\end{aligned}
$$

where $r_i(P) = \max\{r, \max\{\|p_i - p_j\|_2 \mid j \in \mathcal{N}_G(i)\}\}$. Since two neighbors according to G can be arbitrarily far from each other in $\mathbb{R}^d$, we need to modify the definition of the constraint set with the radius $r_i(P)$ to prevent $\mathcal{X}_i$ from becoming empty. Note that if $\|p_i - p_j\|_2 \leq r$ for all $j \in \mathcal{N}_G(i)$, then $r_i(P) = r$ and, therefore, $\mathcal{X}_i = \mathcal{X}_{\text{disk},G}(p_i, P) \cap \overline{B}(p_i, u_{\max})$. It is also worth observing that both $(p_{\text{goal}})_i$ and $\mathcal{X}_i$ change continuously with $(p_1, \dots, p_n)$.

Define the map $\text{fti}_G : (\mathbb{R}^d)^n \to (\mathbb{R}^d)^n$ by

$$\text{fti}_G(p_1, \dots, p_n) = (\text{fti}(p_1, (p_{\text{goal}})_1, \mathcal{X}_1), \dots, \text{fti}(p_n, (p_{\text{goal}})_n, \mathcal{X}_n)).$$

One can think of fti_G as a circumcenter law where the neighboring relationships among the agents never change. Because fti is continuous, and $(p_{\text{goal}})_i$ and $\mathcal{X}_i$, $i \in \{1, \dots, n\}$, change continuously with $(p_1, \dots, p_n)$, we deduce that fti_G is continuous.

We now define a set-valued dynamical system $((\mathbb{R}^d)^n, (\mathbb{R}^d)^n, T)$ through the set-valued map $T : (\mathbb{R}^d)^n \rightrightarrows (\mathbb{R}^d)^n$ given by

$$T(p_1, \dots, p_n) = \{\text{fti}_G(p_1, \dots, p_n) \mid G \text{ is a strongly connected digraph}\}.$$

Note that the evolution of the CRCMCNTR law using a proximity graph such as $\mathcal{G}_{\text{disk}}(r)$ is just one of the multiple evolutions described by this set-valued map. This concludes (a).

Let us now perform (b). To characterize the convergence properties of the set-valued dynamical system, we use the LaSalle Invariance Principle in Theorem 1.21. With the notation of this result, we select $W = (\mathbb{R}^d)^n$. This set is clearly strongly positively invariant for $((\mathbb{R}^d)^n, (\mathbb{R}^d)^n, T)$.

Closedness of the set-valued map. Since fti_G is continuous for each digraph G and there is a finite number of strongly connected digraphs on the vertices $\{1, \dots, n\}$, Exercise E1.9 implies that T is closed.

Common Lyapunov function. Define the function $V_{\text{diam}} : (\mathbb{R}^d)^n \to \mathbb{R}_{\geq 0}$ by

$$V_{\text{diam}}(P) = \max\{\|p_i - p_j\| \mid i, j \in \{1, \dots, n\}\}.$$

With a slight abuse of notation, we denote by $\text{co}(P)$ the convex hull of $\{p_1, \dots, p_n\} \subset \mathbb{R}^d$. Note that $V_{\text{diam}}(P) = \text{diam}(\text{co}(P))$. The function V_{diam} has the following properties:

(i) V_{diam} is continuous and invariant under permutations of its arguments.

(ii) $V_{\text{diam}}(P) = 0$ if and only if $P \in \text{diag}((\mathbb{R}^d)^n)$, where we recall that $\text{diag}((\mathbb{R}^d)^n) = \{(p_1, \dots, p_n) \in (\mathbb{R}^d)^n \mid p^{[i]} = \cdots = p^{[n]} \in \mathbb{R}^d\}$ denotes the diagonal set of $(\mathbb{R}^d)^n$. This fact is an immediate consequence of the fact that, given a set $S \subset (\mathbb{R}^d)^n$, $\text{diam}(\text{co}(S)) = 0$ if and only if S is a singleton.

(iii) V_{diam} is non-increasing along T on $(\mathbb{R}^d)^n$. Consider a finite set of points $S \in \mathbb{F}((\mathbb{R}^d)^n)$ and let $\text{CC}(S)$ be its circumcenter. From Lemma 2.2(i), we have $\text{CC}(S) \in \text{co}(S)$. Therefore, for any strongly connected digraph G, we have that $\text{co}(\text{fti}_G(P)) \subset \text{co}(P)$ for any $P \in (\mathbb{R}^d)^n$. Since for any two sets $S_1, S_2 \subset (\mathbb{R}^d)^n$ such that $\text{co}(S_2) \subset \text{co}(S_2)$ it holds that $V_{\text{diam}}(S_2) \leq V_{\text{diam}}(S_1)$, then $V_{\text{diam}}(\text{fti}_G(P)) \leq V_{\text{diam}}(P)$ for any strongly connected digraph G, which implies that V_{diam} is non-increasing along T on $(\mathbb{R}^d)^n$.

Bounded evolutions. Consider any initial condition $(p_1(0), \dots, p_n(0)) \in (\mathbb{R}^d)^n$. For any strongly connected digraph, G, we have

$$\text{fti}_G(p_1(\ell), \dots, p_n(\ell)) \in \text{co}(p_1(0), \dots, p_n(0)),$$

for all $\ell \in \mathbb{Z}_{\geq 0}$. Therefore, any evolution of the set-valued dynamical system $((\mathbb{R}^d)^n, (\mathbb{R}^d)^n, T)$ is bounded.

Characterization of the invariant set. By the LaSalle Invariance for set-valued dynamical systems in Theorem 1.21, any evolution with initial condition in $W = (\mathbb{R}^d)^n$ approaches the largest weakly positively invariant set M contained in

$$\{P \in (\mathbb{R}^d)^n \mid \exists\, P' \in T(P) \text{ such that } V_{\text{diam}}(P') = V_{\text{diam}}(P)\}.$$

We show that $M = \text{diag}((\mathbb{R}^d)^n)$. Clearly, $\text{diag}((\mathbb{R}^d)^n) \subset M$. To prove the other inclusion, we reason by contradiction. Assume that $P \in M \setminus \text{diag}((\mathbb{R}^d)^n)$ and, therefore, $V_{\text{diam}}(P) > 0$. Let G be a strongly connected digraph and consider $\text{fti}_G(P)$. For each $i \in \{1, \dots, n\}$, we distinguish two cases depending on whether p_i is or is not a vertex of $\text{co}(P)$. If $p_i \notin \text{Ve}(\text{co}(P))$, then Lemma 2.2(i) implies that $\text{fti}(p_i, (p_{\text{goal}})_i, \mathcal{X}_i) \in \text{co}(P) \setminus \text{Ve}(\text{co}(P))$.

If $p_i \in \text{Ve}(\text{co}(P))$, then we must take into consideration the possibility of having more than one agent located at the same point. If the location of all the neighbors of i in the digraph G coincides with p_i, then agent i will not move, and hence $\text{fti}(p_i, (p_{\text{goal}})_i, \mathcal{X}_i) \in \text{Ve}(\text{co}(P))$. However, we can show that the application of fti_G strictly decreases the number of agents located at p_i. Let us denote this number by N_i, that is,

$$N_i = |\{j \in \{1, \dots, n\} \mid p_j = p_i \text{ and } p_j \in \{p_1, \dots, p_n\}\}|.$$

Since the digraph G is strongly connected, there must exist at least an agent located at p_i with a neighbor which is not located at p_i (otherwise, all agents would be at p_i, which is a contradiction). In other words, there exist $i_*, j \in \{1, \dots, n\}$ such that $p_{i_*} = p_i$, $p_j \neq p_i$, and $j \in \mathcal{N}_G(i_*)$. By Lemma 2.2(i), we have that $(p_{\text{goal}})_{i_*} \in \text{co}(P) \setminus \text{Ve}(\text{co}(P))$ and, therefore,

$(p_{\text{goal}})_{i_*} \neq p_{i_*}$. Combining this with the fact that

$$\{p_i\} \cup \{p_j \mid j \in \mathcal{N}_G(i)\} \subset \overline{B}(p_{i_*}, r_{i_*}(P)),$$

we can apply Lemma 2.2(ii) to ensure that $]p_{i_*}, (p_{\text{goal}})_{i_*}[$ has nonempty intersection with $\mathcal{X}_{i_*}$. Therefore, $\text{fti}(p_{i_*}, (p_{\text{goal}})_{i_*}, \mathcal{X}_{i_*}) \in \text{co}(P) \setminus \text{Ve}(\text{co}(P))$, and the number N_i of agents located at p_i decreases at least by one with the application of fti_G.

Next, we show that, after a finite number of steps, no agents remain at the location p_i. Define $N = \max\{N_i \mid p_i \in \text{Ve}(\text{co}(P))\} < n - 1$. Then all agents in the configuration $\text{fti}_{G_1}(\text{fti}_{G_2}(\dots \text{fti}_{G_N}(P)))$ are contained in $\text{co}(P) \setminus \text{Ve}(\text{co}(P))$, for any collection of strongly connected directed graphs $G_1, \dots, G_N$. Therefore, $\text{diam}(\text{co}(\text{fti}_{G_1}(\text{fti}_{G_2}(\dots \text{fti}_{G_N}(P))))) < \text{diam}(\text{co}(P))$, which contradicts the fact that M is weakly invariant.

Point convergence. We have proved that any evolution of $((\mathbb{R}^d)^n, (\mathbb{R}^d)^n, T)$ approaches the set $\text{diag}((\mathbb{R}^d)^n)$. To conclude the proof, let us show that the convergence of each trajectory is to a point, rather than to the diagonal set. Let $\{P(\ell) \mid \ell \in \mathbb{Z}_{\geq 0}\}$ be an evolution of the set-valued dynamical system. Since the sequence is contained in the compact set $\text{co}(P(0))$, there exists a convergent subsequence $\{P(\ell_k) \mid k \in \mathbb{Z}_{\geq 0}\}$, that is, there exists $p \in \mathbb{R}^d$ such that

$$\lim_{k \to +\infty} P(\ell_k) = (p, \dots, p). \tag{4.6.2}$$

Let us show that the whole sequence $\{P(\ell) \mid \ell \in \mathbb{Z}_{\geq 0}\}$ converges to $(p, \dots, p)$. Because of (4.6.2), for any $\varepsilon > 0$, there exists k_0 such that for $k \geq k_0$ one has $\text{co}(P(\ell_k)) \subset \overline{B}(p, \varepsilon/\sqrt{n})$. From this, we deduce that $\text{co}(P(\ell)) \subset \overline{B}(p, \varepsilon/\sqrt{n})$ for all $\ell \geq \ell_{k_0}$, which in turn implies that $\|P(\ell) - (p, \dots, p)\|_2 \leq \varepsilon$ for all $\ell \geq \ell_{k_0}$, as claimed. This concludes (b).

The steps (a) and (b) imply that any evolution of $(\mathcal{S}_{\text{disk}}, \mathcal{CC}_{\text{CRCMCNTR}})$ starting from an initial configuration where $\mathcal{G}_{\text{disk}}(r)$ is connected converges to a point in $\text{diag}((\mathbb{R}^d)^n)$. To conclude the proof of fact (i), we only need to establish that this convergence is in finite time. This last fact is a consequence of Exercise E4.5.

Fact (v) for $(\mathcal{S}_{\text{disk}}, \mathcal{CC}_{\text{CRCMCNTR}})$ is a consequence of facts (i) and (iv).

STEP 2: Facts on $(\mathcal{S}_{\text{LD}}, \mathcal{CC}_{\text{CRCMCNTR}})$. The proof of facts (i), (iv), and (v) for $(\mathcal{S}_{\text{LD}}, \mathcal{CC}_{\text{CRCMCNTR}})$ is analogous to the proof of these facts for the pair $(\mathcal{S}_{\text{disk}}, \mathcal{CC}_{\text{CRCMCNTR}})$, and we leave it to the reader.

STEP 3: Facts on $(\mathcal{S}_{\infty\text{-disk}}, \mathcal{CC}_{\text{PLL-CRCMCNTR}})$. From the expression for the control function of $\mathcal{CC}_{\text{PLL-CRCMCNTR}}$, we deduce that the evolution under $\mathcal{CC}_{\text{PLL-CRCMCNTR}}$ of the robotic network $\mathcal{S}_{\infty\text{-disk}}$ (in d dimensions) can be alternatively described as the evolution under $\mathcal{CC}_{\text{CRCMCNTR}}$ of d robotic networks $\mathcal{S}_{\text{disk}}$ in $\mathbb{R}$ (see Exercise E4.4). Therefore, facts (i), (iv), and (v) for the pair $(\mathcal{S}_{\infty\text{-disk}}, \mathcal{CC}_{\text{PLL-CRCMCNTR}})$ follow from facts (i), (iv), and (v) for the pair $(\mathcal{S}_{\text{disk}}, \mathcal{CC}_{\text{CRCMCNTR}})$. ■

4.6.3 Proof of Theorem 4.17

Proof. Let $P_0 = (p^{[1]}(0), \dots, p^{[n]}(0)) \in \mathbb{R}^n$ denote the initial condition.

Fact (i). For $d = 1$, the connectivity constraints on each agent $i \in \{1, \dots, n\}$ imposed by the constraint set

$$\mathcal{X}_{\text{disk}}(p^{[i]}, \{p_{\text{rcvd}} \mid \text{for all non-null } p_{\text{rcvd}} \in y^{[i]}\}) \tag{4.6.3}$$

are superfluous. In other words, the goal configuration resulting from the evaluation by agent i of the control function of the CRCMCNTR law belongs to the constraint set in (4.6.3). Moreover, the order of the robots on the real line is preserved from one time step to the next. Both observations are a consequence of Exercise E4.3.

Let us first establish the upper bound in fact (i). Consider the case when $\mathcal{G}_{\text{disk}}(r)$ is connected at P_0. Without loss of generality, assume that the agents are ordered from left to right according to their identifier, that is, $p^{[1]}(0) \leq \dots \leq p^{[n]}(0)$. Let $\alpha \in \{3, \dots, n\}$ have the property that agents $\{2, \dots, \alpha - 1\}$ are neighbors of agent 1, and agent α is not. (If, instead, all agents are within an interval of length r, then rendezvous is achieved after one time step, and the upper bound in fact (i) is easily seen to be true.) Figure 4.12 presents an illustration of the definition of α. Note that we can

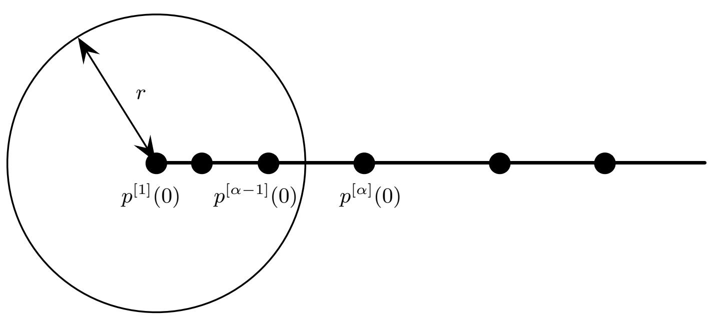


Figure 4.12 The definition of $\alpha \in \{3, \dots, n\}$ for an initial network configuration.

assume that agents $\{2, \dots, \alpha - 1\}$ are also neighbors of agent α. If this is not the case, then those agents that are neighbors of agent 1 and not of agent

α, rendezvous with agent 1 after one time step. At the time instant $\ell = 1$, the new updated positions satisfy

$$p^{[1]}(1) = \frac{p^{[1]}(0) + p^{[\alpha-1]}(0)}{2},$$
$$p^{[\gamma]}(1) \in \left[\frac{p^{[1]}(0) + p^{[\alpha]}(0)}{2}, \frac{p^{[1]}(0) + p^{[\gamma]}(0) + r}{2}\right],$$

for $\gamma \in \{2, \dots, \alpha - 1\}$. These equalities imply that $p^{[1]}(1) - p^{[1]}(0) = \frac{1}{2}\big(p^{[\alpha-1]}(0) - p^{[1]}(0)\big) \leq \frac{1}{2}r$. Analogously, we deduce $p^{[1]}(2) - p^{[1]}(1) \leq \frac{1}{2}r$ and, therefore,

$$p^{[1]}(2) - p^{[1]}(0) \leq r. \tag{4.6.4}$$

On the other hand, from $p^{[1]}(2) \in \big[\frac{1}{2}\big(p^{[1]}(1) + p^{[\alpha-1]}(1)\big), *\big]$ (where the symbol $*$ represents a certain unimportant point in $\mathbb{R}$), we deduce

$$\begin{aligned} p^{[1]}(2) - p^{[1]}(0) &\geq \frac{1}{2}\big(p^{[1]}(1) + p^{[\alpha-1]}(1)\big) - p^{[1]}(0) \\ &\geq \frac{1}{2}\big(p^{[\alpha-1]}(1) - p^{[1]}(0)\big) \geq \frac{1}{2}\Big(\frac{p^{[1]}(0) + p^{[\alpha]}(0)}{2} - p^{[1]}(0)\Big) \\ &= \frac{1}{4}\big(p^{[\alpha]}(0) - p^{[1]}(0)\big) \geq \frac{1}{4}r\,. \end{aligned} \tag{4.6.5}$$

Inequalities (4.6.4) and (4.6.5) mean that, after at most two time steps, agent 1 has traveled a distance greater than $r/4$. In turn, this implies that

$$\frac{1}{r}\operatorname{diam}(\operatorname{co}(P_0)) \leq \operatorname{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{CRCMCNTR}}, P_0) \leq \frac{4}{r}\operatorname{diam}(\operatorname{co}(P_0)).$$

If $\mathcal{G}_{\text{disk}}(r)$ is not connected at P_0, note that along the network evolution, the connected components of the r-disk graph do not change. Using the previous characterization on the distance traveled by the leftmost agent of each connected component in at most two time steps, we deduce that

$$\operatorname{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{CRCMCNTR}}, P_0) \leq \frac{4}{r}\max_{C \in \mathcal{C}(P_0)} \operatorname{diam}(\operatorname{co}(C)),$$

where $\mathcal{C}(P_0)$ denotes the collection of connected components of $\mathcal{G}_{\text{disk}}(r)$ at P_0. The connectedness of each $C \in \mathcal{C}(P_0)$ implies that $\operatorname{diam}(\operatorname{co}(C)) \leq (n-1)r$, and therefore, $\operatorname{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{CRCMCNTR}}) \in O(n)$.

The lower bound in fact (i) is established by considering $P_0 \in \mathbb{R}^n$ such that $p^{[i+1]}(0) - p^{[i]}(0) = r$, $i \in \{1, \dots, n-1\}$. For this configuration, we have $\operatorname{diam}(\operatorname{co}(P_0)) = (n-1)r$ and, therefore, $\operatorname{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{CRCMCNTR}}, P_0) \geq n-1$.

Fact (ii). In the r-limited Delaunay graph, two agents on the line that are at most at a distance r from each other are neighbors if and only if there are no other agents between them. Also, note that the r-limited Delaunay graph and the r-disk graph have the same connected components (cf., Theorem 2.8). An argument similar to the one used in the proof of fact (i) above guarantees that the connectivity constraints imposed by the constraint sets $\mathcal{X}_{\text{disk}}(p^{[i]}, \{p_{\text{rcvd}} \mid \text{for all non-null } p_{\text{rcvd}} \in y^{[i]}\})$ are again superfluous.

Consider first the case when $\mathcal{G}_{\text{LD}}(r)$ is connected at P_0. Note that this is equivalent to $\mathcal{G}_{\text{disk}}(r)$ being connected at P_0. Without loss of generality, assume that the agents are ordered from left to right according to their identifier, that is, $p^{[1]}(0) \leq \cdots \leq p^{[n]}(0)$. The evolution of the network under $\mathcal{CC}_{\text{CRCMCNTR}}$ can then be described as the discrete-time dynamical system

$$\begin{aligned}
p^{[1]}(\ell+1) &= \frac{1}{2}(p^{[1]}(\ell) + p^{[2]}(\ell)), \\
p^{[2]}(\ell+1) &= \frac{1}{2}(p^{[1]}(\ell) + p^{[3]}(\ell)), \\
&\vdots \\
p^{[n-1]}(\ell+1) &= \frac{1}{2}(p^{[n-2]}(\ell) + p^{[n]}(\ell)), \\
p^{[n]}(\ell+1) &= \frac{1}{2}(p^{[n-1]}(\ell) + p^{[n]}(\ell)).
\end{aligned}$$

Note that this evolution respects the ordering of the agents. Equivalently, we can write $P(\ell+1) = A\,P(\ell)$, where $A \in \mathbb{R}^{n\times n}$ is the matrix given by

$$A = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & \dots & \dots & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & \dots & \dots & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 \\
\vdots & & \ddots & \ddots & \ddots & \vdots \\
0 & \dots & \dots & \frac{1}{2} & 0 & \frac{1}{2} \\
0 & \dots & \dots & 0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}.$$

Note that $A = \text{ATrid}_n^+\left(\frac{1}{2}, 0\right)$, as defined in Section 1.6.4. Theorem 1.80(i) implies that, for $P_{\text{ave}} = \frac{1}{n}\mathbf{1}_n^T P_0$, we have that $\lim_{\ell\to+\infty} P(\ell) = P_{\text{ave}}\mathbf{1}_n$, and that the maximum time required for $\|P(\ell) - P_{\text{ave}}\mathbf{1}_n\|_2 \leq \eta\|P_0 - P_{\text{ave}}\mathbf{1}_n\|_2$ (over all initial conditions in $\mathbb{R}^n$) is $\Theta(n^2 \log \eta^{-1})$. (Note that this also implies that agents rendezvous at the location given by the average of their initial positions. In other words, the asymptotic rendezvous position for this case can be expressed in closed form, as opposed to the case with the r-disk graph.)

Next, let us convert the contraction inequality on 2-norms into an appropriate inequality on ∞-norms. Note that $\text{diam}(\text{co}(P_0)) \leq (n-1)r$ because

$\mathcal{G}_{\mathrm{LD}}(r)$ is connected at P_0. Therefore,

$$\|P_0 - P_{\mathrm{ave}}\mathbf{1}\|_\infty = \max_{i\in\{1,\dots,n\}} |p^{[i]}(0) - P_{\mathrm{ave}}| \le |p^{[1]}(0) - p^{[n]}(0)| \le (n-1)r.$$

For ℓ of order $n^2 \log \eta^{-1}$, we use this bound on $\|P_0 - P_{\mathrm{ave}}\mathbf{1}\|_\infty$ and the basic inequalities $\|v\|_\infty \le \|v\|_2 \le \sqrt{n}\|v\|_\infty$ for all $v \in \mathbb{R}^n$, to obtain

$$\begin{aligned}\|P(\ell) - P_{\mathrm{ave}}\mathbf{1}\|_\infty &\le \|P(\ell) - P_{\mathrm{ave}}\mathbf{1}\|_2 \le \eta\|P_0 - P_{\mathrm{ave}}\mathbf{1}\|_2 \\ &\le \eta\sqrt{n}\|P_0 - P_{\mathrm{ave}}\mathbf{1}\|_\infty \le \eta\sqrt{n}(n-1)r.\end{aligned}$$

This means that $(r\varepsilon)$-rendezvous is achieved for $\eta\sqrt{n}(n-1)r = r\varepsilon$, that is, in time $O(n^2 \log \eta^{-1}) = O(n^2 \log(n\varepsilon^{-1}))$.

Next, we show the lower bound. Consider the unit-length eigenvector $\mathbf{v}_n = \sqrt{\frac{2}{n+1}}(\sin\frac{\pi}{n+1}, \dots, \sin\frac{n\pi}{n+1})^T \in \mathbb{R}^n$ of $\mathrm{Trid}_{n-1}(\frac{1}{2}, 0, \frac{1}{2})$ corresponding to the largest singular value $\cos(\frac{\pi}{n})$. For $\mu = \frac{-1}{10\sqrt{2}} r n^{5/2}$, we then define the initial condition

$$P_0 = \mu P_+ \begin{bmatrix} 0 \\ \mathbf{v}_{n-1} \end{bmatrix} \in \mathbb{R}^n.$$

One can show that $p^{[i]}(0) < p^{[i+1]}(0)$ for $i \in \{1, \dots, n-1\}$, that $P_{\mathrm{ave}} = 0$, and that $\max\{p^{[i+1]}(0) - p^{[i]}(0) \mid i \in \{1, \dots, n-1\}\} \le r$. Using Lemma 1.82 and because $\|w\|_\infty \le \|w\|_2 \le \sqrt{n}\|w\|_\infty$ for all $w \in \mathbb{R}^n$, we compute

$$\begin{aligned}\|P_0\|_\infty = \frac{rn^{5/2}}{10\sqrt{2}} \left\| P_+ \begin{bmatrix} 0 \\ \mathbf{v}_{n-1} \end{bmatrix} \right\|_\infty &\ge \frac{rn^2}{10\sqrt{2}} \left\| P_+ \begin{bmatrix} 0 \\ \mathbf{v}_{n-1} \end{bmatrix} \right\|_2 \\ \ge \frac{rn}{10\sqrt{2}} \|\mathbf{v}_{n-1}\|_2 = \frac{rn}{10\sqrt{2}}&.\end{aligned}$$

The trajectory $P(\ell) = (\cos(\frac{\pi}{n}))^\ell P_0$ therefore satisfies

$$\|P(\ell)\|_\infty = \left(\cos\left(\frac{\pi}{n}\right)\right)^\ell \|P_0\|_\infty \ge \frac{rn}{10\sqrt{2}} \left(\cos\left(\frac{\pi}{n}\right)\right)^\ell.$$

Therefore, $\|P(\ell)\|_\infty$ is larger than $\frac{1}{2}r\varepsilon$ so long as $\frac{1}{10\sqrt{2}} n (\cos(\frac{\pi}{n}))^\ell > \frac{1}{2}\varepsilon$, that is, so long as

$$\ell < \frac{\log(\varepsilon^{-1} n) - \log(5\sqrt{2})}{-\log\left(\cos(\frac{\pi}{n})\right)}.$$

In exercise E4.7, the reader is asked to show that the asymptotics of this bound correspond to the lower bound in fact (i).

Now consider the case when $\mathcal{G}_{\mathrm{LD}}(r)$ is not connected at P_0. Note that the connected components do not change along the network evolution. Therefore, the previous reasoning can be applied to each connected component.

Since the number of agents in each connected component is strictly less than n, the time complexity can only but improve. Therefore, we conclude that

$$\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{CRCMCNTR}}) \in \Theta(n^2 \log(n\varepsilon^{-1})).$$

Fact (iii). Recall from the proof of Theorem 4.6.2 that the evolution under $\mathcal{CC}_{\mathrm{PLL\text{-}CRCMCNTR}}$ of the robotic network $\mathcal{S}_{\infty\text{-disk}}$ (in d dimensions) can be alternatively described as the evolution under $\mathcal{CC}_{\mathrm{CRCMCNTR}}$ of d robotic networks $\mathcal{S}_{\mathrm{disk}}$ in $\mathbb{R}$ (see Exercise E4.4). Fact (iii) now follows from fact (i). ■

4.6.4 Proof sketch of Theorem 4.20

Here, we only provide a sketch of the proof of Theorem 4.20. Fact (i) is a consequence of the control function definition of the NONCONVEX CRCMCNTR law in Section 4.3.4 and Lemma 4.11. Fact (ii) follows from the fact that the law $\mathcal{CC}_{\mathrm{NONCONVEX\ CRCMCNTR}}$ (with control magnitude bounds) achieves the ε-rendezvous task $\mathcal{T}_{\varepsilon\text{-rndzvs}}$ and fact (i).

To show that, on the network $\mathcal{S}_{\text{vis-disk}}$, the law $\mathcal{CC}_{\mathrm{NONCONVEX\ CRCMCNTR}}$ (with control magnitude bounds) achieves the ε-rendezvous task $\mathcal{T}_{\varepsilon\text{-rndzvs}}$, one can follow the same overapproximation strategy that we used in the proof of Theorem 4.16, *STEP 1:*, that is,

(a) define a set-valued dynamical system $(Q_\delta^n, Q_\delta^n, T)$ such that the evolutions of $(\mathcal{S}_{\text{vis-disk}}, \mathcal{CC}_{\mathrm{NONCONVEX\ CRCMCNTR}})$ starting from an initial configuration where $\mathcal{G}_{\text{vis-disk},Q_\delta}$ is connected are contained in the set of evolutions of the set-valued dynamical system; and

(b) establish that any evolution of $(Q_\delta^n, Q_\delta^n, T)$ converges to a point in $\mathrm{diag}(Q_\delta^n)$ (note that the point might be different for different evolutions).

We refer to Ganguli et al. (2009) for a detailed development of this proof strategy. Here, we only remark that in order to carry out (b), the proof uses the LaSalle Invariance Principle in Theorem 1.21, with the perimeter of the relative convex hull of a set of points as Lyapunov function.

4.7 EXERCISES

E4.1 **(Maintaining connectivity of sparser networks).** Prove Lemma 4.8.
Hint: *Use Lemma 4.2 and the fact that $\mathcal{G}$ and $\mathcal{G}_{\mathrm{disk}}(r)$ have the same connected components.*

E4.2 **(Maintaining network line-of-sight connectivity).** Prove Lemma 4.11.
Hint: *Use Proposition 4.9.*

E4.3 **(Enforcing range-limited links is unnecessary for the crcmcntr law on $\mathbb{R}$).** Let $\mathcal{P} = \{p_1, \dots, p_n\} \in \mathbb{F}(\mathbb{R})$. For $r \in \mathbb{R}_{>0}$, we work with the r-disk proximity graph $\mathcal{G}_{\text{disk}}(r)$ evaluated at $\mathcal{P}$. Let $i \in \{1, \dots, n\}$ and consider the circumcenter of the set comprised of p_i and of its neighbors:

$$(p_{\text{goal}})_i = \text{CC}(\{p_i\} \cup \mathcal{N}_{\mathcal{G}_{\text{disk}}(r), p_i}(\mathcal{P})).$$

Show that the following hold:

(i) if p_i and p_j are neighbors in $\mathcal{G}_{\text{disk}}(r)$, then $(p_{\text{goal}})_i$ belongs to $\overline{B}(\frac{p_i+p_j}{2}, \frac{r}{2})$;

(ii) if p_i and p_j are neighbors in $\mathcal{G}_{\text{disk}}(r)$ and $p_i \leq p_j$, then $(p_{\text{goal}})_i \leq (p_{\text{goal}})_j$; and

Finally, discuss the implication of (i) and (ii) in the execution of the CRCMCNTR law on the 1-dimensional space $\mathbb{R}$.
Hint: *Express $(p_{\text{goal}})_i$ as a function of the position of the leftmost and rightmost points among the neighbors of p_i.*

E4.4 **(Enforcing range-limited links is unnecessary for the pll-crcmcntr law).** Let $\mathcal{P} = \{p_1, \dots, p_n\} \in \mathbb{F}(\mathbb{R}^d)$ and $r \in \mathbb{R}_{>0}$. For $k \in \{1, \dots, d\}$, denote by $\pi_k : \mathbb{R}^d \to \mathbb{R}$ the projection onto the kth component. Do the following tasks:

(i) Show that p_i and p_j are neighbors in $\mathcal{G}_{\infty\text{-disk}}(r)$ if and only if, for all $k \in \{1, \dots, d\}$, $\pi_k(p_i)$ and $\pi_k(p_j)$ are neighbors in $\mathcal{G}_{\text{disk}}(r)$.

(ii) For $S \subset \mathbb{R}^d$, justify that the parallel circumcenter $\text{PCC}(S) \in \mathbb{R}^d$ of S can be described as

$$\pi_k(\text{PCC}(S)) = \text{CC}(\pi_k(S)), \quad \text{for } k \in \{1, \dots, d\}.$$

(iii) Use (i), (ii), and Exercise E4.3(i) to justify that no constraint is required to maintain connectivity of the ∞-disk graph in the PLL-CRCMCNTR law. In other words, show that if p_i and p_j are neighbors in the proximity graph $\mathcal{G}_{\infty\text{-disk}}(r)$, then also the points $\text{PCC}(\{p_i\} \cup \mathcal{N}_{\mathcal{G}_{\infty\text{-disk}}(r), p_i}(\mathcal{P}))$ and $\text{PCC}(\{p_j\} \cup \mathcal{N}_{\mathcal{G}_{\infty\text{-disk}}(r), p_j}(\mathcal{P}))$ are neighbors in the proximity graph $\mathcal{G}_{\infty\text{-disk}}(r)$.

E4.5 **(Finite-time convergence of the crcmcntr law on $\mathcal{S}_{\text{disk}}$).** For $u_{\max}, r \in \mathbb{R}_{>0}$, let $a = \min\{u_{\max}, \frac{r}{2}\}$. Let $\mathcal{P} = \{p_1, \dots, p_n\} \in \mathbb{F}(\mathbb{R}^d)$, and assume that there exists $p \in \mathbb{R}^d$ such that

$$\{p_1, \dots, p_n\} \subset \overline{B}(p, a).$$

Do the following tasks:

(i) Show that $\mathcal{G}_{\text{disk}}(r)$ evaluated at $\{p_1, \dots, p_n\}$ is the complete graph.

(ii) Justify why $\|p_i - \text{CC}(\{p_1, \dots, p_n\})\|_2 \leq a$, for all $i \in \{1, \dots, n\}$.

(iii) Show that $\text{CC}(\{p_1, \dots, p_n\}) \in \mathcal{X}_{\text{disk}}(p_i, \mathcal{P}) \cap \overline{B}(p_i, u_{\max})$.

(iv) What is the evolution of the pair $(\mathcal{S}_{\text{disk}}, \mathcal{CC}_{\text{CRCMCNTR}})$ (with control magnitude bounds) starting from $(p_1, \dots, p_n)$?

E4.6 **(Variation of the crcmcntr law).** Let $\mathcal{P} = \{p_1, \dots, p_n\} \in \mathbb{F}(\mathbb{R}^d)$. For $r \in \mathbb{R}_{>0}$, we work with the r-disk proximity graph $\mathcal{G}_{\text{disk}}(r)$ evaluated at $\mathcal{P}$. For each $i \in \{1, \dots, n\}$, consider the circumcenter of the set comprised of p_i and of the mid-points with its neighbors:

$$(p_{\text{goal}})_i = \text{CC}\big(\{p_i\} \cup \big\{\frac{p_i + p_j}{2} \mid p_j \in \mathcal{N}_{\mathcal{G}_{\text{disk}}(r),p_i}(\mathcal{P})\big\}\big).$$

Do the following:

(i) Show that if p_i and p_j are neighbors in $\mathcal{G}_{\text{disk}}(r)$, then $(p_{\text{goal}})_i$ and $(p_{\text{goal}})_j$ are neighbors in $\mathcal{G}_{\text{disk}}(r)$.

(ii) Use (i) to design a control and communication law on the network $\mathcal{S}_{\text{disk}}$ in $\mathbb{R}^d$ that, while not enforcing any connectivity constraints, preserves all neighboring relationships in $\mathcal{G}_{\text{disk}}(r)$ and achieves the ε-rendezvous task $\mathcal{T}_{\varepsilon\text{-rndzvs}}$.

(iii) Justify why the law designed in (ii) does not achieve the exact rendezvous task $\mathcal{T}_{\text{rndzvs}}$.

E4.7 **(Asymptotics of the lower bound in Theorem 4.17(ii)).** Show that, as $n \to +\infty$,

$$\frac{\log(\varepsilon^{-1}n) - \log(5\sqrt{2})}{-\log\big(\cos(\frac{\pi}{n})\big)} = \frac{n^2}{\pi^2}\big(\log(\varepsilon^{-1}n) - \log(5\sqrt{2})\big) + O(1).$$

Use this fact to complete the proof of the lower bound in the proof of Theorem 4.17(ii).
Hint: *Use the Taylor series expansion of* $\log(\cos(x))$ *at* $x = 0$.

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